Students' understanding of the concept of limit of a function in vocational high school mathematics

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Abstract: The authors report on a study which used the APOS theory to examine vocational high school students' understanding of limit of functions. The limits of real functions were taught to computer technology, electricity-energy, electronics and automation, construction, and machinery-metal technology programs students at a vocational high school in Kocaeli University in Turkey. It is given the analysis of students' responses to four types of questions on limits of real functions; limits of split-functions, limits at infinity of rational functions, limits of functions not defined at a point, and continuity of real functions. As a main finding of the study, vocational high school students find difficult to understand the limits of real functions. Also, it suggests that this is possibly the result of many students not having appropriate mental structures at the process, object and schema levels.

Keywords: limits of real functions, the Apos theory, vocational high school.

Introduction

APOS (Action ‑ Process ‑ Object ‑ Schema) theory suggests that an individual has to have appropriate mental structures to make sense of a given mathematical concept. The mental structures refer to the likely actions, processes, objects and schema which are required to learn the concept. Research based on this theory requires that for a given concept the likely mental structures need to be detected, and then suitable learning activities should be designed to support the construction of these mental structures in the students' mind (Dubinsky & McDonald, 2001).

APOS theory

The main mental operations for building the mental structures of action, process, object, and schema are called "interiorisation" and "encapsulation" (Dubinsky, 2010; Weller et al., 2003). The descriptions of action, process, object and schema are based on those given by Weller, Arnon and Dubinsky (2009);
Action: A transformation is first conceived as an action, when it is a reaction to stimuli which an individual perceives as external. It requires specific teaching, and the need to perform each step of the transformation explicitly. For example, a student who requires an explicit expression to think about a limit of a function,$\lim_{x\to a} f(x)$, and can do little more than substitute values of $x$close to $a$for the variable in the expression $f(x)$and manipulate it, is considered to have an action understanding of a limit of a function.
Process: As an individual repeats and reflects on an action, it may be interiorized into a mental process. A process is a mental structure that performs the same operation as the action, but wholly in the mind of the individual. Specifically, the individual can imagine performing the transformation without having to execute each step explicitly. For example, an individual with a process understanding of the limit of a function;$\lim_{x\to a} f(x)$; will construct a mental process for values of x close to a and think in terms of inputs, possibly unspecified, and transformations of those inputs to produce outputs.

Object: If one becomes aware of a process as a totality, realises that transformations can act on that totality and can actually construct such transformations (explicitly or in one's imagination), then we say the individual has encapsulated the process into a cognitive object. For example, for the limit of a function concept an individual may confront situations requiring him/her to apply various actions and/or processes. These could include thinking about an operation that takes two functions and produces a new function, such as in

$\lim_{x \to 2^+} \frac{|x-2|}{x-2}$. In order to operate on the one sided-limit of this new function, the process understanding must be encapsulated and converted to an object.

Schema: A mathematical topic often involves many actions, processes, and objects that need to be organised and linked into a coherent framework, called a schema. It is coherent in that it provides an individual with a way of deciding, when presented with a particular mathematical situation, whether the schema applies. For example, the coherence might lie in the understanding that to determine the existence of a limit of a function, $\lim_{x \to a} f(x)$, the following must be considered: input values to the left and right of a, the corresponding output values, and a means of transforming elements of the inputs to elements of the outputs.

The ACE teaching cycle

This pedagogical approach, based on APOS theory and the hypothesis on learning and teaching, is a repeated cycle consisting of three components: (A) activities, (C) classroom discussion, and (E) exercises done outside of class (Asiala, et. al., 1996). The activities, which form the first step of the cycle, are designed to foster the students' development of the mental structures called for by an APOS analysis. In the classroom the teacher guides the students to reflect on the activities and its relation to the mathematical concepts being studied. Students do this by performing mathematical tasks. They discuss their results and listen to explanations, by follow students or the teacher, of the mathematical meanings of what they are working on. The homework exercises are fairly standard problems. They reinforce the knowledge obtained in the activities and classroom discussions. Students apply this knowledge to solve standard problems related to the topic being studied. The implementation of this approach and its effectiveness in helping students make mental constructions and learn mathematics has been reported in several research studies. A summary of early work can be found in Weller et al. (2003).

Literature review

There are many studies on students' understanding of the concept of a limit of a function (e.g. Cornu, 1992; Davis & Vinner, 1986; Li & Tall, 1993; Maharajh, Brijlall, & Govender, 2008; Monaghan, Sun, & Tall, 1994; Tall, 1992; Tall & Vinner, 1981; Williams, 1991). These studies indicate that students have difficulties with the concept of a limit of a function in the context of functions and continuity or series and sequences, and many of the difficulties encountered by students in dealing with other concepts; for example continuity, differentiability and integration; are related to their difficulties with limits.

Some researchers (Cornu, 1992; Sierpińska, 1987) reported that a high percentage of students have a static view of mathematics. Such students can only deal with a very specific calculation that is placed before them. Students with such a view will have difficulties with the limit of a function concept. The term "precept" is used to indicate that mathematical symbolism can ambiguously represent either a process or a concept (Monaghan et al., 1994). So, the symbol $\lim_{x \to a} f(x)$is an example of a procept since it represents the process of getting to a specific value, or the value of the limit of the function itself. However, unlike the procepts of elementary mathematics, where an algorithm can be used to calculate the specific value of the concept, the limit value does not have a universal algorithm that works in all cases. Further, the limit of a function concept is not restricted to a finite computation that gives a definitive answer. This is precisely where the distinction between an action and a process comes in. It could be argued that once a calculation involves an infinite number of steps, it could only be understood through a process conception. A commonly cited difficulty that students have in constructing a process conception of limit of a function is their perception of a limit of a function as something that is actually never attained (Cottrill, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Dubinsky, 2010).

It seems that many students perform poorly because they: (a) are unable to adequately handle information given in symbolic form which represent objects (abstract entities), for example functions, and (b) lack adequate schema or frameworks, which help to organise and link different objects (Maharaj, 2005). Giraldo, Carvalho and Tall (2003) distinguish between a description of a concept, which specifies some properties of that concept and the formal concept definition. Some descriptions commonly employed in the teaching of limits of functions include table of values, graphical and algebraic representations. Individually these involve limitations that do not fully reflect the mathematical situation. The teaching implication is that a variety of representations should be used, and to encourage students to engage with a flexibility of mathematical conceptions of $\lim_{x \to a} f(x)$. The research questions for this study were:

- How should the teaching of the concept of a limit of a function be approached?

- What insights would an APOS analysis of students' understanding of the concept of a limit of a function reveal?

Materials and Method

The participants for this study were 672 vocational high school mathematics students at a university in Turkey in 2012; about %92 of these were first year students. The students were studying a compulsory mathematics course. The aim of the course studied is to introduce students to the fundamental principles, methods, procedures and techniques of mathematics as the language of science. These students attended their lectures for two hours in a week. I was the lecturer to all the students. This was the context for the theoretical analysis of the limit of a function concept.

The key question for a 50 minute lecture session was: When does $\lim_{x\to a} f(x)$ exist? Activities were formulated and these were projected by use of a PC tablet. A reasonable time was given for students to reflect and work on each activity; they were free to discuss with other students sitting beside them and to use the prescribed textbook. While students engaged with the activities I observed how they worked, their difficulties and aspects that required further explanations. These informed my explanations; using a PC tablet; to the class. Another 45 minute session was devoted to activities based on techniques for finding limits of functions, including limits at infinity. The activities and explanations incorporated use of graphical representations to answer questions on limits of functions, including limits of split-functions of the types given in symbolic notation. Activities and classroom discussions were followed by homework exercises, which students had to work on as part of their tutorial requirements. A PC tablet was used to summarise the lecture-room discussions .In their groups they could further discuss the homework exercises with their tutors.

About 3 weeks after the tutorials a multiple choice questions (MCQs) test was administered to 646 students. The questions set were similar to those for the activities and homework. Students were required to first work out the solutions in the space below each question and then to mark their choices on the multiple-choice-question cards. Note that the fourth question is based on the concept of continuity, which incorporates the concept of the existence of a limit of a function. The teaching for the section on continuity was similar to that outlined above, in that the ACE teaching cycle was followed. The options given for each of the MCQs were constructed bearing in mind the APOS levels of mental structures.

Results and Discussion

To represent the analysis, findings and discussion for each of the four questions in a reader friendly format, the following subheadings which describe the type of question are used:

- Limits of split-functions
- Continuity application of split-functions
- Limits of functions not defined at a point
- Limits of rational functions at infinity

Under each of these subheadings the relevant test item and question analysis is given. The question analysis indicating number of student choices and the percentage (correct to one decimal place) for each of the four questions are indicated in Tables 1 to 4 below. In each of these tables "*" denotes the letter of the correct answer. The Omit index gives the number of students who did not mark any of the alternatives, and the Bad index gives the number of students who marked more than one choice.

1. Limits of split-functions

Let $f(x) = \begin{cases} 2x+1; & x>5 \\ 3x-6\,; & x\le 5 \end{cases}$

$$\lim_{x\to 5} f(x) = ?$$

A) 11 B) 9 C) 10 D) does not exist E) none of these

Tablo1: Question 1 analysis of student choices (N = 646)

	A	B	C	D*	E	Omit index	Bad index
Student choices	95	102	21	332	33	59	4
Percentage	14,7	15,7	3,2	51,3	5,1	9,1	0,6

Question 1 is based on finding the limit of a split-function f (x), as $x$approaches a value in the domain where the function is split. The question analysis in Table 1, using the totals for choices A and B, indicates that 197 students conceptualized $\lim_{x\to 5} f(x)$to be the same as one of the one-sided $\lim_{x\to 5^+} f(x)$or $\lim_{x\to 5^-} f(x)$. In the APOS framework, using the genetic decomposition I arrived at, this means that those students' mental constructions were at best at the action level. This suggests that for evaluation of limits of split-functions, approximately %30 of the students had mental constructions developed up to the action level. The numbers for choice C suggests that a total of 21 students (approximately %3) had no idea of the basic technique for finding the limit of a split-function given in algebraic form; that is when the split-function is expressed in symbolic notation. A possible reason for 218 students selecting choices A, B or C is that they did not fully understand the concept of a split function. This implies that it seems that approximately %33 of the students did not understand the concept of a split-function when such a function is represented in algebraic form.

Table 1 also indicates that 332 students marked the correct choice for Question 1. In the context of my decomposition this suggests that for evaluation of limits, of the type of split-functions under discussion, approximately %51 of the students had mental constructions developed up to the object level. Table 1 indicates that 33 students chose option E. This suggests approximately %5 of the students could have made calculation errors; and if this is accepted; their mental constructions were probably functioning at the object level. So it seems that approximately %56 of the students had appropriate mental structures in place for some sort of effective schema to evaluate the limit of a split-function f(x) given in symbolic form, as $x$approaches $a$value in the domain where the function is split. Table 1 also indicates that a large number of students (59, about %9) did not indicate any choice. One of the reasons for this is that they did not have any idea of how to work out the solution of such questions.

2. Limits at infinity of rational functions

$$\lim_{x\to\infty} \frac{4+7x-5x^2}{-1-10x^2} = ?$$

A) 1/2 B) ∞ C) 0 D) -1/4 E) none of these

Tablo2: Question 2 analysis of student choices (N = 646)

	A*	B	C	D	E	Omit index	Bad index
Student choices	302	136	41	34	102	25	6
Percentage	46,7	21	6,3	5,2	15,7	3,8	0,9

Question 2 is based on finding the limit at infinity of rational functions; the case where the polynomials in the numerator and denominator are of the same degree. Table 2 implies that the 177 students (approximately %27); those who chose options B and C; had no appropriate schema to deal with finding the limits at infinity of rational functions. Noting that $-1/4$ is the reciprocal of $-4/1$, Table 2 suggests that the 34 students (approximately %5) had mental constructions not even at the action level. Table 2also implies that at least 302students (approximately %47) had appropriately developed schema to deal with problems based on finding the limits at infinity of rational functions; in particular the case where the polynomials in the numerator and denominator are of the same degree. The 102 students (approximately %16) who chose option E could have made calculation errors; and if this is accepted; their mental constructions probably incorporated appropriate schema for finding limits of rational functions, at infinity. So it seems that at most approximately %62 of the students had mental structures appropriately developed to some sort of effective schema for finding limits of rational functions, at infinity; in particular the case where the degree of the polynomial in the numerator is equal to that of the polynomial in the denominator.

3. Limits of functions not defined at a point

$$\lim_{x \to 25} \frac{\sqrt{x} - 5}{x - 25} = ?$$

A) 1/5 B) 0 C) 1/10 D) ∞ E) -∞

Tablo3: Question3 analysis of student choices (N = 646)

	A	B	C*	D	E	Omit index	Bad index
Student choices	83	212	185	72	30	61	3
Percentage	12,8	32,8	28,6	11,1	4,6	9,4	0,4

Question 3 is based on the evaluation of limits of functions not defined at a point. The limit of the function cannot be found by finding the corresponding function value. The technique here is to express the function in factorised form $\frac{\sqrt{x}-5}{(\sqrt{x}+5)(\sqrt{x}-5)}$ noting that$x \neq 25$ simplifies this to$\frac{1}{(\sqrt{x}+5)}$ and then findinglim$_{x\to 25}\frac{1}{(\sqrt{x}+5)}$. Noting that$\frac{-5}{-25} = \frac{1}{5}$, Table 3 suggests that the 83 students (approximately %13) had mental constructions which were not even at an action level. The distracters given as options B, D and E were designed for responses at a limited action level. If this is accepted then Table 3 suggests that 314 students (approximately %49) possibly used mental constructions at some sort of action level. The 185 students (approximately %29) who marked the correct answer, conceptualized $\lim_{x\to 25}\frac{\sqrt{x}-5}{x-25}$as an object. Further, using this object it seems they had appropriate schema to deal with the relevant imbedded and implied information. Table 3 suggests that 61 students (approximately %9) did not have any idea of how to work out the solution of such questions. So, it seems that at least %22 of the students had mental constructions not developed to any of the levels indicated in APOS Theory.

4. Continuity application of split-functions

Let $f(x) = \begin{cases} \frac{3x^2+5x-2}{x+2}; & x \neq -2 \\ mx + 1\,; & x = -2 \end{cases}$

What is the value of m that will make the function f continuous?

A) 2 B) 4 C) -2 D) 3 E) none of these

Tablo4: Question3 analysis of student choices (N = 646)

	A	B*	C	D	E	Omit index	Bad index
Student choices	96	166	25	63	61	228	7
Percentage	14,8	25,6	3,8	9,7	9,4	35,2	1,0

An analysis of Question 4 reveals that the point of discontinuity occurs at$x = -2$. This is the crucial observation from the structure of the given split-function. The question is based on finding an unknown coefficient of one of the functions; in a split-function; which will make the entire function continuous on the interval$(-\infty, \infty)$. This requires continuity at$x = -2$, which implies that the condition $\lim_{x\to -2} f(x) = f(-2)$ must be satisfied. The type of mental conception required here involves the formulation of $\lim_{x\to -2} f(x) = f(-2)$ as an equation which should be treated as an object. Further the successful use of this equation depends on an appropriately developed schema. This must incorporate conceptualization of split-functions represented in symbolic form, as objects.

Note that in Question 4, if $x \neq -2$then$\frac{3x^2+5x-2}{x+2}$. So options A, C and D deal with distracters which were arrived at from this structure. If this is accepted then the question analysis in Table 4; using the totals for options A, C and D; indicates that 184 students (approximately %28) possibly had mental constructions which were not even at the action level. Table 4 also suggests that 166 students (approximately %26) possibly had appropriately developed schema for applications on continuity of split-functions represented in symbolic form. Since 61 students chose option E, this suggests approximately %9 of the students could have made calculation errors. If this is accepted then their mental constructions probably incorporated appropriate schema for applications on continuity of split functions represented in symbolic form. So it seems that approximately %35 of the students had appropriate mental structures in place for some sort of effective schema for applications on continuity of split-functions represented in symbolic form. Table 4 also indicates that a large number of students 228 (approximately %35) did not indicate any choice. One of the reasons for this is that they did not have any idea of how to work out the solution of such questions.

The above analyses indicated that the types of MCQs used basically gave information on the highest potential mental structure levels of those students, according to APOS Theory.

Conclusions

Useful insight into the relevant mental structures towards which teaching should focus was revealed by the APOS genetic decomposition of the limit of a function concept. The findings of this study confirmed that the limit of a function concept is one that students find difficult to understand, and suggests that this is possibly the result of many students not having appropriate mental structures at the process, object and schema levels. It seems that my genetic decomposition was adequate. However, my reflections on the teaching design indicated that more time needs to be devoted to helping students develop the mental structures at the process, object and schema levels. This implies that teaching should focus on (1) verbal and graphical approaches to finding limits of functions; including split-functions in symbolic form, (2) unpacking of structures given in symbolic form, and (3) modeling possible schema. A graphical approach should facilitate the development of mental structures at the process and object levels, while a focus on symbolic structures should aid object conceptions. If schemas organize and link the relevant actions, processes and objects then this should be a part of the teaching. The impact of such a focus on teaching will require further research.

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