MINIMIZING THE ENERGY OF THE VELOCITY VECTOR FIELD OF CURVE IN R^3

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Abstract: The present paper considers all the unit-speed curve segments between two fixed points p and q in R^3. It obtain a condition for the critical curve of the problem of minimizing the energy of the velocity vector field among the family of all curves from p to q. It show that the condition can be expressed in terms of the curvature functions.

Keywords: Energy, Energy of a unit vector field, Sasaki metric

Introduction

The volume of unit vector fields has been studied by (Gluck and Ziller, 1986, Johnson, 1988, Higuchi, Kay and Wood, 2001) among other scientists. They define the volume of unit vector field X as the volume of the submanifold of the unit tangent bundle defined by $X(M)$. In (Wood, 1997), the energy of a unit vector field on a Riemannian manifold M is defined as the energy of the mapping $X: M \to T^1M$, where the unit tangent bundle T^1M is equipped with the restriction of the Sasaki metric on TM.

Generally, every geometric problem about curves can be solved using the curves'Frenet vectors field. Therefore, in (Altın, 2011), we focus on the curve C instead of the manifold M. For a given curve C, with a pair of parametric unit speeds (I, α) in a space R^n, on which we take a fixed point $a \in I$, we denote Frenet frames at the points $\alpha(a)$ and $\alpha(s)$ by $\{V_1(\alpha(a)), \dots, V_r(\alpha(a))\}$ and $\{V_1(\alpha(s)), \dots, V_r(\alpha(s))\}$ respectively. We calculate the energy of the Frenet vectors fields as well as the angle between the vectors $V_i(\alpha(a))$ and $V_i(\alpha(s))$, where $1 \leq i \leq r$. We observed that both energy and angle depend on the curvature functions of the curve C.

In this paper, we choose two points p and q in R^3. We obtain a condition for the critical curve of the problem of minimizing the energy of the velocity vector field among the family of all curves from p to q. We also prove that this condition can be expressed in terms of the curvature functions. For example the condition is realized for curves whose curvature functions is constant. An example is also provided to show that the curvature of the curve is linear.

Definition 1.1 A curve segment is the portion of a curve defined in a closed interval, (O'Neill 1966).

Theorem 1.1.(Frenet formulas) If $\alpha: I \to R^3$ is a unit speed curve with curvature $\kappa > 0$ and torsion τ, then

$$T' = \kappa N,$$
$$N' = -\kappa T + \tau B,$$
$$B' = -\tau N$$

Where $\{T, N, B\}$ is the Frenet frame on α (O'Neill 1966).

Proposition 1.1 The connection map $K: T(T^1M) \to T^1M$ verifies the following conditions.
1) $\pi \circ K = \pi \circ d\pi$ and $\pi \circ K = \pi \circ \tilde{\pi}$, where $\tilde{\pi}: T(T^1M) \to T^1M$ is the tangent bundle projection and $\pi: T^1M \to M$ is the bundle projection.

2) For $\omega \in T_xM$ and a section $\xi: M \to T^1M$, we have

$$K(d\xi(\omega)) = \nabla_\omega \xi.$$

Where ∇ is the Levi-Civita covariant derivative (Chacón , Naveira and Weston,2001).

Definition 1.2. For $\eta_{1,}\ \ \eta_1 \in T_\xi(T^1M)$ define

$$g_s(\eta_{1,}\ \ \eta_1) = < d\pi(\eta_1), d\pi(\eta_2) > + < K(\eta_1), K(\eta_2) >. \quad (1)$$

This gives a Riemannian metric on TM. Recall that g_s is called the Sasaki metric. The metric g_s makes the projection $\pi: T^1M \to M$ a Riemannian submersion (Chacón , Naveira and Weston,2001).

Definition 1.3. The energy of a differentiable map $f: (\mathrm{M}, <,>) \to (N, h)$ between Riemannian manifolds is given by

$$\varepsilon(f) = \frac{1}{2}\int_M \left(\sum_{a=1}^{n} h(df(e_a), df(e_a)\right) v \tag{2}$$

where v is the canonical volume form in M and $\{e_a\}$ is a local basis of the tangent space (Chacón , Naveira and Weston,2001 and Wood, 1997).

A Condition on Minimizing Energy of the Velocity Vector Field of a Curve in R^3

The following theorem characterizes a critical point of the energy of the velocity vector field of a curve in R^3

Theorem 2.1. Let α be unit speed curve in R^3 and α(a) = p, α(b) = q. Let us consider the collection of all curves segments from p to q in R^3 . If the energy of the velocity vector of α along one segment is less than that along any other segment, then the following equation is valid

$$\int_a^b \lambda(s)\kappa(s)\kappa'(s)ds = 0 \tag{3}$$

where κ is the curvature function and λ is the real-valued function n [a, b].

Proof. Let $\alpha: I \to R^3$ be a unit speed curve in R^3 and [a, b]⊂ I, α(a)= p, α(b) = q. There exists a real-valued function λ on [a, b], λ(s)=(s-a)(b-s), λ(a)=λ(b)=0 and λ(s)≠ 0 for all s∈(a, b). Let {T, N, B} be the Frenet frame field on α and

$$\lambda(s)T(s) = \big(v_1(s),\ v_2(s),\ v_3(s)\big), \qquad v_i: [a.b] \to R. \tag{4}$$

Let the collection of curves be

$$\alpha^k(s) = \big(\alpha_1(s)+k\ \ v_1(s),\ \alpha_2(s)+k\ \ v_2(s),\ \alpha_3(s)+k\ \ v_3(s)\big) \textit{ for sufficiently small k.} \tag{5}$$

For k=0, α^0(s)=α(s) and λ(a)=λ(b)=0, we have v_i(a)= v_i(b)=0 . $1 \le i \le 3$ and $\alpha^k(a) = p,\ \ \alpha^k(b) = q.$ These results show that α^k is the curve segment from p to q.

Assume this collection $\alpha^k(s)$ =α(s, k) for all curves. The expression for the energy of the vector field T_k of α^k from p to q becomes $\mathcal{E}(T_k)$.

Now, let TC_k be the tangent bundle. So we have $T_k: C_k \to TC_k$, where $TC_k = \cup_{t\in I} T_{\alpha^k(t)}C_k$, $C_k = \alpha^k(I)$ and $T_{\alpha^k(t)}C_k$ denotes generated by T_k. Let π: $TC_k \to C_k$ be the bundle projection. By using equation (2) we calculate the energy of T_k as

$$\mathcal{E}(T_k) = \frac{1}{2}\int_a^b g_S(d\,T_k(T_k(\alpha(s,k)), dT_k(T_k\big(\alpha(s,k)\big))ds \tag{6}$$

where ds is the differential arc length. From (1) we have

$$g_S(dT_k(T_k),\ dT_k(T_k)) = \ < d\pi(\ dT_k(T_k)),\ d\pi(\ dT_k(T_k)) > + < K(dT_k(T_k)),\ K(dT_k(T_k) >.$$

Since T_k is a section, we have $d(\pi) \circ d(T_k) = d(\pi \circ T_k) = d\big(id_{C_k}\big) = id_{TC_k}$. By Proposition 1.1, we also have that

$$K\big(dT_k(T_k)\big) = \nabla_{T_k}T_k = T_k' = \frac{\partial T_k}{\partial s},$$

giving

$$g_S(dT_k(T_k),\ dT_k(T_k)) = \ < T_k,\ T_k > + < T'_k,\ T'_k >.$$

Using these results in (6) we get

$$\mathcal{E}(T_k) = \frac{1}{2} \int_a^b (< T_k, T_k > + < T'_k, T'_k >) ds \tag{7}$$

Where $T_k = \frac{1}{w(s,k)} \frac{d\alpha}{\partial s}(s, k)$; $w(s,k) = \sqrt{< \frac{d\alpha}{\partial s}(s,k), \frac{d\alpha}{\partial s}(s,k) >}$, Suppose that $\mathcal{E}(T_k)$ is such a minimized energy for any "k" α(s, k) with α(s,0)=α(s). We calculate $\frac{\partial \mathcal{E}(T_k)}{\partial k}$ and evaluate at k=0. If $\mathcal{E}(T_k)$ is a minimizing energy, then k=0 should be a critical point of $\mathcal{E}(T_k)$. Supposing that $\frac{\partial \mathcal{E}(T_k)}{\partial k}|_{k=0} = \frac{\partial \mathcal{E}(T_0)}{\partial k} = 0$, from (7) we obtain:

$$\frac{\partial \mathcal{E}(T_k)}{\partial k} = \frac{\partial}{\partial k} \left[\frac{1}{2} \int_a^b (< T_k, T_k > + < T'_k, T'_k >) ds \right] = \frac{1}{2} \int_a^b \frac{\partial}{\partial k} \left[< T_k, T_k > + < \frac{\partial T_k}{\partial s}, \frac{\partial T_k}{\partial s} > \right] ds.$$

Since $< T_k, T_k > = 1$ we have $\frac{\partial}{\partial k} < T_k, T_k > = 0$ and we get

$$\frac{\partial \mathcal{E}(T_k)}{\partial k} = \frac{1}{2} \int_a^b \frac{\partial}{\partial k} < \frac{\partial T_k}{\partial s}, \frac{\partial T_k}{\partial s} > ds = \int_a^b (< \frac{\partial^2 T_k}{\partial s \partial k}, \frac{\partial T_k}{\partial s} > ds. \tag{8}$$

We can write

$$\frac{\partial}{\partial s} < \frac{\partial T_k}{\partial k}, \frac{\partial T_k}{\partial s} > = < \frac{\partial^2 T_k}{\partial s \partial k}, \frac{\partial T_k}{\partial s} > + < \frac{\partial T_k}{\partial k}, \frac{\partial^2 T_k}{\partial s^2} >.$$

Thus, we can deduce,

$$< \frac{\partial^2 T_k}{\partial s \partial k}, \frac{\partial T_k}{\partial s} > = \frac{\partial}{\partial s} < \frac{\partial T_k}{\partial k}, \frac{\partial T_k}{\partial s} > - < \frac{\partial T_k}{\partial k}, \frac{\partial^2 T_k}{\partial s^2} >. \tag{9}$$

Substituting (9) in (8), for, k=0,

$$\frac{\partial \mathcal{E}(T_0)}{\partial k} = \int_a^b \left[\frac{\partial}{\partial s} < \frac{\partial T_k}{\partial k}(s,0), \frac{\partial T_k}{\partial s}(s,0) > - < \frac{\partial T_k}{\partial k}(s,0), \frac{\partial^2 T_k}{\partial s^2}(s,0) > \right] ds$$

and

$$\frac{\partial \mathcal{E}(T_0)}{\partial k} = < \frac{\partial T_k}{\partial k}(s,0), \frac{\partial T_k}{\partial s}(s,0) > |_a^b - \int_a^b < \frac{\partial T_k}{\partial k}(s,0), \frac{\partial^2 T_k}{\partial s^2}(s,0) > ds \tag{10}$$

From (4) and (5), we obtain,

$$\frac{\partial \alpha}{\partial k}(s, k) = \lambda(s)\ T_k(s) \tag{11}$$

and

$$\frac{\partial \alpha}{\partial k}(s, 0) = \alpha'(s) = T_k(s, 0) \tag{12}$$

Now we calculate the partial derivatives of (12) with respect to s and k; using Frenet formulas, we get

$$\frac{\partial T_k}{\partial s}(s,0) = \frac{\partial^2 \alpha}{\partial s^2}(s,0) = \alpha''(s) = T'(s) = \kappa(s) N(s) \tag{13}$$

and

$$\frac{\partial T_k}{\partial k}(s,k) = \frac{\partial^2 \alpha}{\partial s \partial k}(s, k) = \frac{\partial^2 \alpha}{\partial k \partial s}(s, k).$$

From (11), we have

$$\frac{\partial T_k}{\partial k}(s,k)|_{k=0} = \frac{\partial T}{\partial k}(s,0) = \lambda'(s) T(s) + \lambda(s) \kappa(s) N(s). \tag{14}$$

It follows from (13) and (14) that

$$< \frac{\partial T_k}{\partial k}(s,0), \frac{\partial T_k}{\partial s}(s,0) > = \lambda(s) \kappa^2(s).$$

Considering the candidate function $\lambda(a) = \lambda(b) = 0$, we get:

$$< \frac{\partial T_k}{\partial k}(s,0), \quad \frac{\partial T_k}{\partial s}(s,0) > |_a^b = \lambda(b)\kappa^2(b) - \lambda(a)\kappa^2(a) = 0 \tag{15}$$

From (13), we get

$$\frac{\partial^2 T_k}{\partial s^2}(s,0) = -\kappa^2(s)T(s) + \kappa'(s)N(s) + \kappa(s)\tau(s)B(s) \tag{16}$$

Therefore, (14) and (16) gives

$$< \frac{\partial T_k}{\partial k}(s,0), \frac{\partial^2 T_k}{\partial s^2}(s,0) > = -\lambda'(s)\kappa^2(s) + \lambda(s)\kappa(s)\kappa'(s) = [-\lambda(s)\kappa^2(s)]' + 3\lambda(s)\kappa(s)\kappa'(s). \tag{17}$$

Substituting (15) and (17) in (10), yields

$$\frac{\partial \mathcal{E}(T_k)}{\partial k}|_{k=0} = \frac{\partial \mathcal{E}(T_0)}{\partial k} = -\int_a^b([-\lambda(s)\kappa^2(s)]' + 3\lambda(s)\kappa(s)\kappa'(s))ds = 0$$

and

$$\frac{\partial \mathcal{E}(T_0)}{\partial k} = [-\ \lambda(s)\kappa^2(s)]|_a^b - 3\int_a^b \lambda(s)\kappa(s)\kappa'(s)ds = 0$$

We are looking the candidate function $\lambda(a) = \lambda(b) = 0$, which given $[-\lambda(s)\kappa^2(s)]|_a^b = 0$ and

$$\frac{\partial \mathcal{E}(T_0)}{\partial k} = -3\int_a^b \lambda(s)\kappa(s)\kappa'(s)ds = 0.$$

This completes the proof of the theorem. Any path that minimizes the energy function $\mathcal{E}(T_k)$ must satisfy equation (3). Note that the condition is necessary, but not sufficient; not every function that satisfies (3) will produce minimal energy. If α is geodesic then it will satisfy equation (3). The following is provided as an example. This
example will also demonstrate that the curvature of the curve is linear, given the aforementioned conditions.

Example. Let $\alpha: I \to R^3$, [0,1]⊂ I, α(0)= p, α(1) = q. If we can choose $\lambda: [0,1] \to R$, λ(s)=s(1-s), λ(0)=0, λ(1)=0 and λ(s)≠ 0 for all s∈(0, 1). Let the curvature function of α be $\kappa(s) = cs + d$ where c and d are real numbers. Using equation (3), we have

$$\frac{\partial \mathcal{E}(T_0)}{\partial k} = \int_0^1 \lambda(s)\kappa(s)\kappa'(s)ds = c(c+2d) = 0.$$

If c=0 then κ constant, or c=-2d.

References

Altın, A. (2011),. On the Energy and the Pseudo Angle of a Frenet Vectors Fields in R_v^n. *Ukrainian Mathematical journal*, Vol. 63, No 6, (pp.833-839).
Chacón, P.M. and Naveira, A.M. (2004), Corrected Energy of Distributions on Riemannian Manifold. *Osaka Journal Mathematics*,Vol 41 (pp. 97-105).
Chacón, P.M. and Naveira, A.M. and Weston , J.M. (2001), *On the Energy of Distributions, with Application to the Quarternionic Hopf Fibration*.. Monatshefte fûr Mathematik, Vol.133,(pp. 281-294).
Gluck. H. and Ziller, W. (1986), On the volume of the unit vector fields on the three sphere. *Comment Math. Helv*. Vol 61, (pp.177-192).
Higuchi, A., Kay, B. S. and Wood, C. M. (2001), The energy of unit vector fields on the 3-sphere *Journal of Geometry and Physics*. Vol 37, (pp. 137-155).
Johnson, D.L. (1988), Volume of flows, Proc. *Amer. Math. Soc*.Vol.104, (pp. 923-932).
O'Neill, B. (1966). Elementary Diffrential Geometry. *Academic Press Inc*.
Wood, C. M. (1997), On the Energy of a Unit Vector Field. *Geometrae Dedicata*. Vol.64, (pp.319-330).