NEIGHBOR INTEGRITY OF HARARY GRAPHS

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Abstract: A network is a series of interconnected centers and transmission paths. A system like transportation, communication, computer, logistics, etc., constitutes a network. Failure of the links between stations, break down the centers, software faults on the network, hardware malfunctions, transmission errors that may occur in various centers affect the quality of service to be received over a network, more importantly, it causes a long interruption. In such a situation, to know the vulnerability, the resistance of the network to disruption of operation after the failure of certain stations or links until the system break down completely, is becoming important. A network can be modelled by a graph representing the centers by vertices and the links between the centers by edges. Various vulnerability parameters were defined in graph theory to study the vulnerability of the networks. Connectivity, integrity, toughness, tenacity, scattering number and rupture degree are some of the vulnerability parameters defined in graph theory. But in spy networks, if a spy or a station is revealed, then the adjacent stations cannot be trusted. Neighbor integrity is a vulnerability parameter that considers the neighborhoods and hence can be applied to the spy networks. The neighbor integrity of a graph G is defined to be $NI(G) = \min_{S\subseteq V(G)}\{|S| + c(G/S)\}$ where S is any vertex subversion strategy of G, and c(G/S) is the order of the largest connected component of G/S.

In this study, we deal with the problem of computing the neighbor integrity of Harary graphs which are the graphs having maximum possible connectivity with the minimum number of edges and hence many researchers are interested in studying its stability properties.

Keywords: Graph Theory, vulnerability, neighbor integrity, Harary graphs.

Introducton

A network consists of centers and lines which connect the centers. A network can be divided into different networks or it can be formed by many sub networks. Failure of / break down the links between centers, failure of the centers, software faults on the network, hardware malfunctions, transmission errors that may ocur in various centers affect performance of a network more importantly. They can cause an interruption on the received service over a network for a long time. The resistance of the network to disruption of operation after the failure of certain stations or links until performance of the network completely stops is called vulnerability. To learn about the vulnerability of the network is very important for the network designers and network analysts. It provides the construction of the network to be designed both in the most appropriate way possible against the threat and in a manner which provides the reconstruction of the network exposed to a damage in a short time. There are different network topologies such as path, tree, star topology and each network topology has advantages and disadvantages. When a network topology need to be modelled mathematically, usage of graph theory emerges as one of the ways of modelling. In a network topology, a network can be modelled by a graph representing the centers by vertices and the links between the centers by edges. Various vulnerability parameters were defined in graph theory to study the vulnerability of the networks. These parameters can be evaluated by using the number of the elements that are not working, the number of the sub networks, and the number of elements in the remaining largest network that can still mutually communicate. Connectivity (Harary, 1969), integrity (Barefoot, 1987), toughness (Goddard, 1990), tenacity (Cozzens et al., 1995), scattering number (Jung, 1978) and rupture degree (Li, 2005) are some of the vulnerability parameters defined in graph theory. Some informations about the vulnerability of the network modelled by graphs can be obtained by using these graph parameters.

The basis of these parameters is the connectivity which is the minimum number of centers that must be removed to disconnect the network. Network designers and network analysts want to design a network which is more reliable or less vulnerable. So they want maximum possible connectivity whereas they want minimum number of edges for the lowest cost network.

For any fixed integers m and n such that $n \geq m + 1$, Harary constructed the class of graphs $H_{m,n}$ which are m-connected with the minimum number of edges on n vertices. So Harary graphs have the maximum possible connectivity with the minimum number of edges and hence many researchers are interested in studying its stability properties (Harary, 1962). Three cases are followed to create Harary graphs:

Case 1: If m is even, let $m = 2k$, then $H_{m,n}$ has vertices $0,1,2,\dots,n-1$, and two vertices i and j are adjacent if and only if $|i-j| \le k$, where the addition is taken modulo n.

Case 2: If m is odd $(m > 1)$ and n is even, $let\ m = 2k+1$ $(k > 0)$, then $H_{m,n}$ is constructed by first drawing $H_{2k,n}$ and then adding edges joining vertex i to vertex $i + \frac{n}{2}$ for $1 \le i \le \frac{n}{2}$.

Case 3: If m is odd $(m > 1)$ and n is odd, let $m = 2k+1$ $(k > 0)$, then $H_{2k+1,n}$ is constructed by first drawing $H_{2k,n}$ and then adding edges joining the vertex i to $i + \frac{n+1}{2}$ for $0 \le i \le \frac{(n-1)}{2}$.

Note that under this definition, the vertex 0 is adjacent to both the vertices $\frac{(n+1)}{2}$ and $\frac{(n-1)}{2}$. Again note that all vertices of $H_{m,n}$ have degree m except for the vertex 0, which has degree $m+1$.

In particular, if a vertex loses its function in the security system or in spy networks then adjacent vertices become disfunctional in the same way. With this in mind, many neighbor vulnerability parameters were determined. One such parameter is the neighbor integrity. The concept of neighbor integrity was introduced by Cozzens and Wu in (Cozzens et al., 1996).

Let G be a simple graph without loops and multiple edges and let u be any vertex in G. The set $N(u) = \{v \in V(G); v \ne u, v \text{ and } u \text{ are adjacent}\}$ is the open neighborhood of u and $N[u] = \{u\} \cup N(u)$ denotes the closed neighborhood of u. A vertex u in G is said to be subverted if the closed neighborhood of u, $N[u]$, is removed from G. A set of vertices $S = \{u_1, u_2, u_3, \dots, u_m\}$ is called a vertex subversion strategy of G if each of the vertices in S has been subverted from G. If S has been subverted from the graph G, then the survival subgraph is disconnected, a clique or an empty graph. The survival subgraph is denoted by G/S. The neighbor integrity of a graph G is defined to be

$$NI(G) = \min_{S \subseteq V(G)} \{|S| + c(G/S)\}$$

where S is any vertex subversion strategy of G and $c(G/S)$ is the maximum order of the components of G/S (Cozzens et al., 1996).

Neighbor Integrity of Harary Graphs

In this section, we investigate lower and upper bounds of the neighbor integrity for the three cases of the Harary graphs.

Theorem 1. Let $H_{m,n}$ be a Harary graph with $m = 2k$ and $n \ge 3$. Then the neighbor integrity of $H_{m,n}$ is

a) If $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$, then

$$NI(H_{m,n}) = \begin{cases} \lceil 2\sqrt{n} \rceil - (2k+1), & \text{if } 1 \le k \le \dfrac{\sqrt{n}-1}{2} \\ \left\lceil \dfrac{n}{2k+1} \right\rceil, & \text{otherwise.} \end{cases}$$

b) If $k > \left\lfloor \frac{n}{2} \right\rfloor$, then $NI(H_{m,n}) = 1$.

Proof. It is obvious that $H_{m,n} \cong C_n^k$. Thus we have $NI(H_{m,n}) = NI(C_n^k)$.

a) For $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$ we get $NI(H_{m,n}) = NI(C_n^k) = \begin{cases} \lceil 2\sqrt{n} \rceil - (2k+1), & \text{if } 1 \le k \le \frac{\sqrt{n}-1}{2} \\ \left\lceil \frac{n}{2k+1} \right\rceil, & \text{otherwise.} \end{cases}$

since it was proved by Cozzens and Wu in (Cozzens, 1998).

b) For $k > \left\lfloor \frac{n}{2} \right\rfloor$, $diam(C_n) = \left\lfloor \frac{n}{2} \right\rfloor \le k$. Thus we get $C_n^k \cong K_n$. Therefore we obtain

$$NI(H_{m,n}) = NI(C_n^k) = NI(K_n) = 1.$$

Theorem 2. Let $H_{m,n}$ be a Harary graph with $m = 2k + 1$ and n even, and let $n \geq (4k + 2)(2k + 2)$ and $k \geq 2$. Then the neighbor integrity of $H_{m,n}$ is

$$2\left\lceil\sqrt{n-(2k+2)}\right\rceil - (2k+1) \leq NI(H_{m,n}) \leq \gamma(H_{m,n}).$$

Proof. Let S be a subversion strategy of $H_{m,n}$ and let $|S| = r$.

If r vertices are removed from $H_{m,n}$, then we have at least $r - 1$ components and $c(H_{m,n}/S) \geq \frac{n-(2k+2)r}{r-1}$. Hence

$$|S| + c(H_{m,n}/S) \geq r + \frac{n-(2k+2)r}{r-1} \text{ and}$$

$$NI(H_{m,n}) \geq \min_{r}\left\{r-(2k+2)+\frac{n-(2k+2)}{r-1}\right\}.$$

The function $f(r) = r - (2k+2) + \frac{n-(2k+2)}{r-1}$ takes its minimum value at $r = \sqrt{n-(2k+2)} + 1$. Consequently, we obtain

$$NI(H_{m,n}) \geq 2\sqrt{n-(2k+2)} - (2k+1).$$

As the neighbor integrity is integer valued, we round this up to get a lower bound and get

$$NI(H_{m,n}) \geq \lceil 2\sqrt{(n-(2k+2))}\rceil - (2k+1).$$

Fort he upper bound, let S be the minimum dominating set of $H_{m,n}$ and the domination number be $\gamma(H_{m,n})$. Then $N[S] = V(G)$ and therefore we have $c(H_{m,n}/S) = 0$. Since according to the definition we have $NI(G) \leq |S| + c(G/S)$, we obtain an upper bound as $NI(G) \leq \gamma(H_{m,n})$.

Theorem 3. Let $H_{m,n}$ be a Harary graph with $m = 2k + 1$ and n odd, and let $n \geq (4k + 5)(2k + 2)$ and $k \geq 2$. Then the neighbor integrity of $H_{m,n}$ is

$$2\left\lceil\sqrt{n-(2k+3)}\right\rceil - (2k+1) \leq NI(H_{m,n}) \leq \gamma(H_{m,n}).$$

Proof.. Let S be a subversion strategy of $H_{m,n}$ and let $|S| = r$.

If r vertices are removed from $H_{m,n}$, then we get at least $r - 1$ components and $c(H_{m,n}/S) \geq r + \frac{n-1-(2k+2)r}{r-1}$. Hence we obtain

$$|S| + c(H_{m,n}/S) \geq r + \frac{n-1-(2k+2)r}{r-1} \text{ and}$$

$$NI(H_{m,n}) \geq \min_{r}\left\{r-(2k+2)+\frac{n-(2k+3)}{r-1}\right\}.$$

The function $f(r) = r - (2k+2) + \frac{n-(2k+3)}{r-1}$ takes its minimum value at $r = \sqrt{n-(2k+3)} + 1$. Consequently we get

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$$NI(H_{m,n}) \geq 2\left\lceil\sqrt{n-(2k+3)}\right\rceil - (2k+1).$$

For the upper bound, the proof is very similar to that of the previous theorem.

Conclusion

Connectivity is a basic measure used to determine the vulnerability of a graph. And it is equal to the minimum number of vertices removed from the graph to make the graph disconnected. We always prefer a stable and durable network which has maximum connectivity. Increase in the connectivity also requires an increase in the number of edges but we prefer to use minimum number of edges for a lowest cost network. Ultimately, we have to balance the cost and stability. Harary graphs which have the maximum possible connectivity with the minimum number of edges provide this balance. In this paper we obtained lower and upper bounds for the neighbor integrity of the three cases of the Harary graphs.

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