AN ANALYTIC VELOCITY AND ACCELERATION ANALYSIS OF A PLANAR PARALLEL 3-RRR MECHANISM

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Abstract: This paper focusses on the velocity and acceleration analysis of a planar parallel kinematic chain by an analytic method. The velocity analysis reveals that there are poses with either no pole configuration or an infinite number of pole configurations. These poses are called singular or twice singular, respectively. It turns out that in general the singular poses are those where the cranks need to reverse the rotation in order to perform the full motion. At twice-singular poses, bifurcations can take place. The analytic method reduces the local velocity and acceleration analysis to systems of linear equations. Their rules for solvability confirm again the results on singular and twice singular poses. Both can be geometrically characterized by the concurrency of triples of lines. The analytic and algorithmic treatment of the global constrained motion leads to an algebraic problem of degree 6.

Keywords: Planar mechanism, planar parallel 3-RRR-robot, velocity analysis, acceleration analysis.

INTRODUCTION

The planar parallel kinematic chain under consideration is a 3-RRR robot as given by Can & Stachel (2014), however with synchronous drives. Therefore, it is in fact a mechanism; we call it, in short, a 3-RRR mechanism. It has with eight links $\Sigma_0, \ldots, \Sigma_7$ and nine revolute joints A_0, B_0, C_0, A_1,..., C_2 with the following properties (see Figure 1):

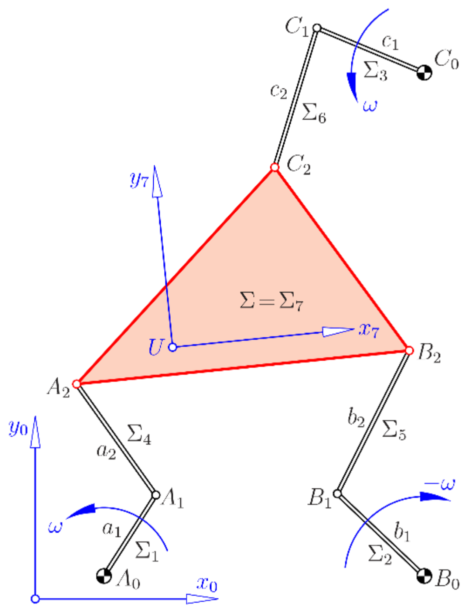

Figure 1. A planar parallel 3-RRR-robot with simultaneously driven cranks

1) There are three driving cranks $A_0A_1 \subset \Sigma_1$, $B_0B_1 \subset \Sigma_2$, and $C_0C_1 \subset \Sigma_3$. They rotate with the same angular velocity ω about the respective anchor points A_0, B_0 and C_0, all fixed in the frame Σ_0. The links Σ_1 and Σ_3 rotate counter-clockwise; Σ_2 rotates either counter-clockwise (direct mechanism) or clockwise (indirect mechanism).

2) The bars $A_1A_2 \subset \Sigma_4$, $B_1B_2 \subset \Sigma_5$ and $C_1C_2 \subset \Sigma_6$ connect the active cranks with the moving frame Σ_7. The points A_2, B_2, C_2 are attached to Σ_7.

3) The lengths of the cranks are denoted by $a_1 = \overline{A_0A_1}$, $b_1 = \overline{B_0B_1}$ and $c_1 = \overline{C_0C_1}$. The bars' lengths are $a_2 = \overline{A_1A_2}$, $b_2 = \overline{B_1B_2}$ and $c_2 = \overline{C_1C_2}$.

General 3-RRR robots control the moving plane via three dyads, i.e., via three RRR mechanisms $A_0A_1A_2$, $B_0B_1B_2$ and $C_0C_1C_2$ by defining the angles of rotation of the active links A_0A_1, B_0B_1 and C_0C_1. Many papers deal with such planar robots, which can be seen as forerunners of spatial parallel robots. Here is a short survey on references:
The forward and reverse kinematics of planar parallel 3-RRR robots has been studied, e.g., by Dijksman (1976), Staicu (2008), Can & Stachel (2014) or by Mahmoodinia, Kamali & Akbarzadeh (2008). Reference Kamali & Akbarzadeh (2011) focuses on the Dynamics of such robots. The singularity analysis is addressed, e.g., by Mohammadi, Zsombor-Murray & Angeles (1995), Bonev & Gosselin (2001), Yang, Chen & Chen (2002), Bonev, Zlatanov & Gosselin (2003) or Di Gregorio (2009). Reference Cha, Lasky & Velinsky (2009) treats the problem of avoiding singularities by the insertion of a redundant parameter. The behavior of the motion in the neighborhood of singularities is studied, e.g., by Voglewede & Ebert-Uphoff (2002). Paper Stan, Maties, Balan & Lapusan (2007) focusses on an algorithmic work space analysis of planar parallel 3-RRR robots.

References on planar parallel 3-RRR robots with synchronously coupled drives are Chen, Angeles & Theingi and Can & Stachel, which were published 2003 and 2014 respectively. The first one focuses mainly on the influence of the driving velocities on the singularities, and in the second one particular attention is laid on a mechanism, i.e., on the case of equal angular velocities for all three active links.

ANALYTICAL GEARS ANALYSIS

In order to describe a movement of the moving space Σ_i with respect to (`w.r.t.', in brief) the fixed space Σ_0 in the d-dimensional Euclidean space $\mathbb{E}^d$, we need Cartesian coordinate systems in both spaces. Then, there is at any instant a transformation between the coordinate vector $\mathbf{x}$ of any point X attached to the moving space Σ_i and the coordinate vector $\mathbf{x}_0$ of the point $X_0 \in \Sigma_0$ which is instantaneously coincident with X. This vector equation is of the type

$$\mathbf{x}_0 = \mathbf{u}_0 + A\mathbf{x}.$$

The matrix $A \in \mathbb{R}^{dxd}$ is directly orthogonal, i.e.

$$AA^T = I_d \text{ und } \det A(t) = 1$$

where $I_d \in \mathbb{R}^{dxd}$ is the identity matrix. Vector $\mathbf{u}_0$ is the coordinate vector of the origin of the moving coordinate frame w.r.t. the coordinate frame in the fixed space Σ_0. We speak in the following "point $\mathbf{x}$" instead of "point X with the coordinate vector $\mathbf{x}$".

Let a twice continuously differentiable one-parameter (=constrained) motion Σ_i/Σ_0 be given. This means, for a parameter t traversing an interval $I \subset \mathbb{R}^d$, there are twofold continuously differentiable functions

$$I \longrightarrow \mathbb{R}^d,\ t \longrightarrow \mathbf{u}_0(t) \ \text{ and } \ I \longrightarrow \mathbb{R}^{dxd},\ t \longrightarrow A(t),$$

such that

$$\Sigma_i/\Sigma_0\colon \mathbf{x}_0 = \mathbf{u}_0(t) + A(t)\mathbf{x} \text{ with } A(t)A(t)^T = I_d,\ det\,A(t) = 1. \tag{1}$$

For any constant vector $\mathbf{x} \in \mathbb{R}^d$ the function $\mathbf{x}_0(t)$ describes in the fixed coordinate frame of Σ_0 the parametrized trajectory of the point $\mathbf{x}$ which is attached to the moving space Σ_i. The parameter t is usually assumed as time.

Velocities

The first derivative $\dot{\mathbf{x}}_0 = \frac{d\mathbf{x}_0}{dt}$ at each instant gives the fixed coordinates of the instantaneous velocity vector $\mathbf{v}_x^{i0}$of the point $\mathbf{x}$ in Σ_i, i.e., the vector of the *vehicular velocity* $\mathbf{v}^f$. When, in addition, a relative movement $\mathbf{x} = \mathbf{x}(t)$ of the point $\mathbf{x}$ w.r.t. Σ_i, is admitted then follows after differentiation

$$\mathbf{v}_\mathrm{x}^{i0} = \dot{\mathbf{x}}_0 = \left(\dot{\mathbf{u}}_0 + \dot{A}\mathbf{x}\right) + A\dot{\mathbf{x}}.$$

The last term yields the fixed coordinates of the vector $\mathbf{v}_0^r$ of the *relative velocity* $\mathbf{v}^r$. The vector on the left-hand side represents the *absolute velocity* $\mathbf{v}_0^a$. That makes altogether

$$\mathbf{v}_0^a = \mathbf{v}_0^f + \mathbf{v}_0^r. \tag{2}$$

We replace in the expression $\mathbf{v}_0^f = \dot{\mathbf{u}}_0 + \dot{A}\mathbf{x}$ for the fixed coordinates of the vehicular velocity vector the coordinate vector $\mathbf{x}$ w.r.t. Σ_i by its instantaneous coordinate vector $\mathbf{x}_0$ w.r.t. Σ_0
according to

$$\mathbf{x} = A^T(\mathbf{x}_0 - \mathbf{u}_0).$$

Thus, we obtain

$$\mathbf{v}_x^{i0} = \mathbf{v}_0^f = \dot{\mathbf{u}}_0 - \dot{A}A^T\mathbf{u}_0 + \dot{A}A^T\mathbf{x}_0.$$

Now we summarize the summands**,** which are independent of $\mathbf{x}_0$ in

$$\mathbf{v}_0 = \dot{\mathbf{u}}_0 - \dot{A}A^T\mathbf{u}_0. \tag{3}$$

Furthermore, we note that for orthogonal matrix A the product $\dot{A}A^T$ is skew-symmetric, since of $A(t)A(t)^T = I_d = const.$ follows by differentiation

$$\dot{A}A^T + A\dot{A}^T = \dot{A}A^T + \left(\dot{A}A^T\right)^T = \dot{I}_d = O_d$$

with $O_d \in \mathbb{R}^{dxd}$ as the zero matrix. Hence, $S = \dot{A}A^T$ satisfies

$$S^T = \left(\dot{A}A^T\right)^T = -\dot{A}A^T = -S.$$

Finally, we obtain for the vehicular velocity

$$\mathbf{v}_x^{i0} = \mathbf{v}_0 + S\mathbf{x}_0, \text{ where } S^T = -S, \tag{4}$$

with $\mathbf{x}_0$ as instantaneous coordinates of the moving point $\mathbf{x} \in \Sigma_i$ and $\mathbf{v}_0$ as a vector of the vehicular velocity of the origin of Σ_i, both with respect to the fixed coordinate frame in Σ_0.

Theorem 1. At any given instant of a constrained motion Σ_i/Σ_0 in $\mathbb{E}^d$, there is a vector $\mathbf{v}_0$ and a skew-symmetric matrix S such that, w.r.t. the fixed coordinate in Σ_0, the vehicular velocity $\mathbf{v}_\mathbf{x}^{i0}$ of the moving point $\mathbf{x} \in \Sigma_i$ with instantaneous coordinates $\mathbf{x}_0$ satisfies

$$\mathbf{v}_\mathbf{x}^{i0} = \mathbf{v}_0 + S\mathbf{x}_0.$$

As the difference of the vehicular velocity vectors of two points $\mathbf{p}, \mathbf{q} \in \Sigma_i$ we obtain the "*velocity vector of* $\mathbf{q}$ *about* $\mathbf{p}$"

$$\mathbf{v}_{\mathbf{qp}} = \mathbf{v}_\mathbf{q} - \mathbf{v}_\mathbf{p} = S(\mathbf{q} - \mathbf{p}).$$

Since, because of the skew-symmetry of S, *the quadratic form* $\mathbf{x}^T S\mathbf{x}$ is the zero form, we obtain

$$(\mathbf{q} - \mathbf{p}).\left(\mathbf{v}_\mathbf{q} - \mathbf{v}_\mathbf{p}\right) = (\mathbf{q} - \mathbf{p})^T\, S\, (\mathbf{q} - \mathbf{p}) = 0.$$

This means, the vector $\mathbf{v}_{\mathbf{qp}}$ is orthogonal to the line connecting the points $\mathbf{p}$ and $\mathbf{q}$. Thus, we have proved

Theorem 2. (Projection theorem) For any two points $\mathbf{p}, \mathbf{q}$ of the same moving system Σ_i the vehicular velocity vectors $\mathbf{v}_\mathbf{p}^{i0}$ and $\mathbf{v}_\mathbf{q}^{i0}$ have equal components in direction of the connecting line, i.e., in direction of $\mathbf{q} - \mathbf{p}$ (see Figure 2).

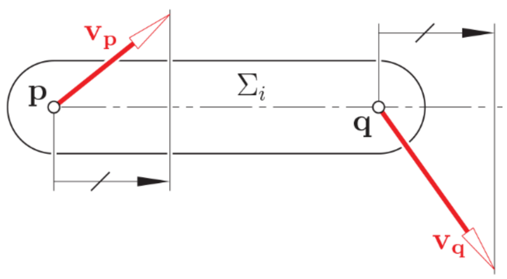

Figure 2. Projection theorem

Remark 1. In case of planar constrained motions, this theorem is equivalent to the so-called graphical *rotated velocities method* from Wunderlich 1970, p. 23.

Accelerations

The second derivative of the parameterized trajectory $\mathbf{x}_0(t)$ of $\mathbf{x} \in \Sigma_i$ gives the acceleration vector

$$\mathbf{a}_x^{i0} = \ddot{\mathbf{x}}_0 = \frac{d^2\mathbf{x}_0}{dt^2}.$$

From $\dot{\mathbf{x}}_0 = \dot{\mathbf{u}}_0 + \dot{A}\mathbf{x} + A\dot{\mathbf{x}}$ we obtain

$$\ddot{\mathbf{x}}_0 = (\ddot{\mathbf{u}}_0 + \ddot{A}\mathbf{x}) + 2\dot{A}\dot{\mathbf{x}} + A\ddot{\mathbf{x}}.$$

The last term gives the fixed coordinates $\mathbf{a}_0^r$ of the vector $\mathbf{a}^r$ of the *relative acceleration*, thus the acceleration of the point $\mathbf{x}$ relative to the moving space Σ_i. The expression in brackets arises at acceleration if $\dot{\mathbf{x}} = \mathbf{0}$; this component is called *vehicular acceleration* $\mathbf{a}_0^f$. The remaining summand

$$\mathbf{a}_0^c = 2\dot{A}\dot{\mathbf{x}} = 2\dot{A}\mathbf{v}^r$$

yields the *Coriolis acceleration*. In total, we obtain as the vector of the *absolute acceleration*

$$\mathbf{a}_0^a = \mathbf{a}_0^f + \mathbf{a}_0^c + \mathbf{a}_0^r. \tag{5}$$

Now we focus on the vehicular acceleration. We replace the moving coordinates of the point $\mathbf{x}$ by its fixed coordinates according to $\mathbf{x} = A^T(\mathbf{x}_0 - \mathbf{u}_0)$ and obtain

$$\mathbf{a}_0^f = \ddot{\mathbf{u}}_0 + \ddot{A}A^T(\mathbf{x}_0 - \mathbf{u}_0) = (\ddot{\mathbf{u}}_0 - \ddot{A}A^T\mathbf{u}_0) + \ddot{A}A^T\mathbf{x}_0. \tag{6}$$

The first term represents the vehicular acceleration of a point in Σ_i, which currently coincides with the origin of the fixed frame ($\mathbf{x}_0 = \mathbf{0}$). Differentiation of the skew-symmetric matrix $S = \dot{A}A^T = -S^T$ yields, $\dot{S} = \ddot{A}A^T + \dot{A}\dot{A}^T$, hence

$$\ddot{A}A^T = \dot{S} - \dot{A}\dot{A}^T = \dot{S} - (\dot{A}A^T)(A\dot{A}^T) = \dot{S} - SS^T = \dot{S} + S^2.$$

Theorem 3. At any instant of a constrained motion Σ_i/Σ_0 in $\mathbb{E}^d$, there is a vector $\mathbf{a}_0$ and a skew-symmetric matrix S, the derivation of the matrix provided in Theorem 1, such that for the vehicular acceleration $\mathbf{a}_x^{i0}$ of any point attached to the moving space Σ_i satisfies in the fixed coordinate frame

$$\mathbf{a}_x^{i0} = \mathbf{a_0} + (\dot{S} + S^2)\mathbf{x}_0.$$

Remark 2. In the special case $d = 2$, the matrix $\dot{S} + S^2$ is a scalar multiple of an orthogonal matrix, because

$$S = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix} \text{ is } \dot{S} + S^2 = \begin{pmatrix} -s^2 & \dot{s} \\ -\dot{s} & -s^2 \end{pmatrix}.$$

The same holds for the sum $I_2 + \dot{S} + S^2$ with the identity matrix.

This is the basis for the second theorem of Burmester: In any graphic representation, there is an orientation preserving similarity between the points in the moving plane Σ_i and the tips of the corresponding acceleration vectors, when protracted from a fixed point. Similarly, the second Theorem of Mehmke states that there is also a direct similarity between the points in the moving plane and the tips of the acceleration vectors when represented as arrows with the respective points as initial points.

The difference of acceleration vectors of two points, $\mathbf{q}$, $\mathbf{p} \in \Sigma_i$ gives the "*acceleration vector of* $\mathbf{q}$ *about* $\mathbf{p}$". According to eq. (6) we obtain

$$\mathbf{a_{qp}} = \mathbf{a_q} - \mathbf{a_p} = (\dot{S} + S^2)(\mathbf{q} - \mathbf{p}) = \dot{S}(\mathbf{q} - \mathbf{p}) + S\mathbf{v_{qp}}.$$

This results in a decomposition of $\mathbf{a_{qp}}$ into two orthogonal components

$$\mathbf{a_{qp}} = \mathbf{a_{qp}^t} + \mathbf{a_{qp}^n}.$$

Because of the skew-symmetry of $\dot{S}$, the first term $\mathbf{a_{qp}^n} = \dot{S}(\mathbf{q} - \mathbf{p})$ is orthogonal to the connection of two points. The second component $\mathbf{a_{qp}^n}$ is parallel to $(\mathbf{q} - \mathbf{p})$.

By virtue of $S^2 = -S^T S$, the component of $\mathbf{a_{qp}}$ in direction of $(\mathbf{p}-\mathbf{q})$ has the signed lenth

$$\frac{(\mathbf{p}-\mathbf{q})^T(\dot{S}+S^2)(\mathbf{q}-\mathbf{p})}{\|(\mathbf{q}-\mathbf{p})\|} = \frac{[(\mathbf{q}-\mathbf{p})^T S^T][S(\mathbf{q}-\mathbf{p})]}{\|(\mathbf{q}-\mathbf{p})\|} = \frac{\mathbf{v}_{\mathbf{qp}}^T \mathbf{v}_{\mathbf{qp}}}{\|(\mathbf{q}-\mathbf{p})\|} = \frac{\|\mathbf{v}_{\mathbf{qp}}\|^2}{\|(\mathbf{q}-\mathbf{p})\|}$$

This explains the relation between the vehicular accelerations of the points **p** and **q** of Σ_i, is displayed in Figure 3.

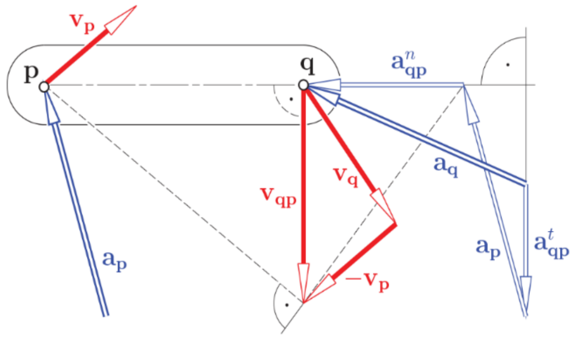

Figure 3. Construction of vehicular acceleration by virtue of $\mathbf{a_{qp}} = \mathbf{a}_{\mathbf{qp}}^t + \mathbf{a}_{\mathbf{qp}}^n$.

Theorem 4. For any two points **p**, **q** of the same moving space Σ_i the difference of their vehicular acceleration vectors $\mathbf{a}_{\mathbf{p}}^{i0}$ and $\mathbf{a}_{\mathbf{q}}^{i0}$ has the representation

$$\mathbf{a_{qp}} = \mathbf{a_q} - \mathbf{a_p} = (\dot{S}+S^2)(\mathbf{q}-\mathbf{p}) = \mathbf{a}_{\mathbf{qp}}^t + \mathbf{a}_{\mathbf{qp}}^n,$$

where $\mathbf{a}_{\mathbf{qp}}^t$ is normal and $\mathbf{a}_{\mathbf{qp}}^n$ is parallel to $(\mathbf{q}-\mathbf{p})$. Furthermore (see Figure 3)

$$\mathbf{a}_{\mathbf{qp}}^n = -\frac{\|\mathbf{v_{qp}}\|^2}{\|(\mathbf{q}-\mathbf{p})\|^2}(\mathbf{q}-\mathbf{p}), \text{ hence } \|\mathbf{a}_{\mathbf{qp}}^n\| = \frac{\|\mathbf{v_{qp}}\|^2}{\|(\mathbf{q}-\mathbf{p})\|}.$$

ANALYTIC VELOCITY ANALYSIS OF THE MECHANISM

We now specify the results on the velocities and accelerations of constrained motions in $\mathbb{E}^d$ to the case $d = 2$ and analyze the mechanism, which has been introduced in section 1 (Figure 1). By Theorem 1, the distribution of the vehicular velocity vectors of the points $\mathbf{x} \in \Sigma_7$ defined by a vector $\mathbf{v_0} \in \mathbb{R}^2$ and a skew-symmetric matrix $S \in \mathbb{R}^{2x2}$. We set:

$$\mathbf{v_0} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & x_3 \\ -x_3 & 0 \end{pmatrix}. \qquad (7)$$

Let us identify all points A_i, B_j and C_k involved at the mechanism with their respective instant coordinate vectors $\mathbf{a}_i$, $\mathbf{b}_j$ and $\mathbf{c}_k$, w.r.t. the coordinate frame in Σ_0. Then the fixed coordinates the velocity vectors of $\mathbf{a}_2$, $\mathbf{b}_2$, $\mathbf{c}_2 \in \Sigma_7$ are

$$\mathbf{v}_{\mathbf{a}_2} = \mathbf{v_0} + S\mathbf{a}_2, \mathbf{v}_{\mathbf{b}_2} = \mathbf{v_0} + S\mathbf{b}_2, \mathbf{v}_{\mathbf{c}_2} = \mathbf{v_0} + S\mathbf{c}_2. \qquad (8)$$

The given angular velocities ± ω of the cranks imply that the velocity vectors of their end points $\mathbf{a}_1$, $\mathbf{b}_1$ and $\mathbf{c}_1$ are

$$\mathbf{v}_{\mathbf{a}_1} = \omega(\mathbf{a}_1 - \mathbf{a}_0)^\perp = \omega D(\mathbf{a}_1 - \mathbf{a}_0),$$
$$\mathbf{v}_{\mathbf{b}_1} = \pm\omega(\mathbf{b}_1 - \mathbf{b}_0)^\perp = \pm\omega D(\mathbf{b}_1 - \mathbf{b}_0), \text{ with } D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
$$\mathbf{v}_{\mathbf{a}_1} = \omega(\mathbf{c}_1 - \mathbf{c}_0)^\perp = \omega D(\mathbf{c}_1 - \mathbf{c}_0) \qquad (9)$$

Here, the symbol ⊥ indicates a positive $\pi/2$-rotation; D is the corresponding (skew-symmetric) matrix. The Projection Theorem 2, applied to the arms, leads to

$$(\mathbf{a}_2 - \mathbf{a}_1)(\mathbf{v}_{\mathbf{a}_2} - \mathbf{v}_{\mathbf{a}_1}) = 0, (\mathbf{b}_2 - \mathbf{b}_1)(\mathbf{v}_{\mathbf{b}_2} - \mathbf{v}_{\mathbf{b}_1}) = 0, (\mathbf{c}_2 - \mathbf{c}_1)(\mathbf{v}_{\mathbf{c}_2} - \mathbf{v}_{\mathbf{c}_1}) = 0. \qquad (10)$$

The substitution of the values from eqs. (8) and (9), therein yields a system of linear equations $M\,\mathbf{x} = \mathbf{s}$ for unknowns (x_1, x_2, x_3) which according to eq. (7) determine the instantaneous velocity of the moving system Σ_7.

There are three cases to distinguish: This linear system of equations has

(i) *a unique solution*, i.e., the rank of the coefficient matrix M is equal 3; or

(ii) *no solution*, i.e., the rank of M is smaller than 3 and also smaller than the rank of the extended coefficient matrix $M_{ext} = (M \mid \mathbf{s})$; or

(iii) *more than one solution*, i.e., the rank of M is smaller than 3 and equal to the rank of M_{ext}.

In order to figure out respective kinematic meaning of these three cases, we introduce the following terms.

Definition 1.
1. A pose of the mechanism is called *singular* if for vanishing driving velocities, i.e., for $\omega = 0$, at least one of the involved $\Sigma_i, i \in \{4, \dots, 7\}$, is infinitesimally movable.
2. A pose of the mechanism is called twice singular if for given driving velocities $\pm\omega$ the local degree of freedom of one of the involved systems is ≥ 2.

Remark 3. The local degree of freedom of any system Σ_i equals the dimension of the vector space of possible velocities - independent of the driving velocity. The local degree of freedom of any mechanism is the maximum of the local degrees of freedom of all involved systems.

Obviously, at a non-singular pose we must have a unique solution for the velocity distribution of Σ_7, i.e., case (i), while at twice singular poses we must meet case (iii).

Theorem 5. Let any pose of the mechanism given. For the velocity distribution of Σ_7, the components (x_1, x_2) of $\mathbf{v_0}$ and non-trivial entry x_3 in the skew-symmetric matrix S solve a system $M\,\mathbf{x} = \mathbf{s}$ of linear equations with

$$M = \begin{pmatrix} (\mathbf{a}_1 - \mathbf{a}_2)^{\perp} & det(\mathbf{a}_1, \mathbf{a}_2) \\ (\mathbf{b}_1 - \mathbf{b}_2)^{\perp} & det(\mathbf{b}_1, \mathbf{b}_2) \\ (\mathbf{c}_1 - \mathbf{c}_2)^{\perp} & det(\mathbf{c}_1, \mathbf{c}_2) \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} \omega\, det(\mathbf{a}_2 - \mathbf{a}_1, \mathbf{a}_1 - \mathbf{a}_0) \\ \pm\,\omega\, det(\mathbf{b}_2 - \mathbf{b}_1, \mathbf{b}_1 - \mathbf{b}_0) \\ \omega\, det(\mathbf{c}_2 - \mathbf{c}_1, \mathbf{c}_1 - \mathbf{c}_0) \end{pmatrix}. \tag{11}$$

1. If the rank of the coefficient matrix M is then < 3 then the position is *singular*.
2. The system has more than one solution if and only if the pose is *twice singular*.

Proof. After plugging eqs. (8) and (9) into eq. (10), we obtain, -*we write only the second equation*-

$$(\mathbf{b}_2 - \mathbf{b}_1)^{\mathrm{T}}\,[\mathbf{v_0} + S\mathbf{b}_2 \pm \omega\, D\,(\mathbf{b}_1 - \mathbf{b}_0)\,] = 0.$$

This equation together with the other two contains bilinear forms with the skew-symmetric coefficient matrices S and D. Their coordinate expressions reveal

$$\mathbf{u}^{\mathrm{T}} S\mathbf{v} = x_3(u_x u_y - u_y u_x) = x_3\, det(\mathbf{u}, \mathbf{v}), \quad \mathbf{u}^{\mathrm{T}} D\mathbf{v} = -det(\mathbf{u}, \mathbf{v}). \tag{12}$$

We can therefore write the second equation of eq. (10) as

$$(\mathbf{b}_2 - \mathbf{b}_1)^{\mathrm{T}} \binom{x_1}{x_2} + x_3\, det(\mathbf{b}_2, \mathbf{b}_1) = \pm\omega\, det(\mathbf{b}_2 - \mathbf{b}_1, \mathbf{b}_1 - \mathbf{b}_0) \tag{13}$$

since

$$det(\mathbf{b}_2 - \mathbf{b}_1, \mathbf{b}_2) = -det(\mathbf{b}_1, \mathbf{b}_2) = det(\mathbf{b}_2, \mathbf{b}_1).$$

Now we are looking for geometric characterizations of the cases (ii) and (iii).

1. Case (ii): The coefficients of x_1, x_2 and x_3 in the linear equation (13) are -up to the order and the signs- identical with the coefficients in the equation of the line connecting $\mathbf{b}_1$ and $\mathbf{b}_2$, hence, of the carrier line of the arm B_1B_2 since in homogeneous notation holds

$$\mathbf{b}_1 \times \mathbf{b}_2 = \begin{pmatrix} 1 \\ b_{1x} \\ b_{1y} \end{pmatrix} \times \begin{pmatrix} 1 \\ b_{2x} \\ b_{2y} \end{pmatrix} = \begin{pmatrix} det(\mathbf{b}_1, \mathbf{b}_2) \\ b_{1y} - b_{2y} \\ b_{2x} - b_{1x} \end{pmatrix}.$$

Thus, any rank deficiency of the matrix M in eq. (11) is equivalent to the linear dependence of three linear equations, i.e., to the existence of a common point of the lines spanned by the arms A_1A_2, B_1B_2 and C_1C_2.

2. Case (iii): With the help of eq. (13) we analyze the extended 3 × 4 coefficient matrix

$$M_{ext} = \left(\begin{array}{cc|c} (\mathbf{a}_1-\mathbf{a}_2)^\perp & det(\mathbf{a}_1,\mathbf{a}_2) & \omega\, det(\mathbf{a}_2-\mathbf{a}_1,\mathbf{a}_1-\mathbf{a}_0) \\ (\mathbf{b}_1-\mathbf{b}_2)^\perp & det(\mathbf{b}_1,\mathbf{b}_2) & \pm\,\omega\, det(\mathbf{b}_2-\mathbf{b}_1,\mathbf{b}_1-\mathbf{b}_0) \\ (\mathbf{c}_1-\mathbf{c}_2)^\perp & det(\mathbf{c}_1,\mathbf{c}_2) & \omega\, det(\mathbf{c}_2-\mathbf{c}_1,\mathbf{c}_1-\mathbf{c}_0) \end{array}\right), \tag{14}$$

and compare the second row with the coefficients in the equation of the line g_2 through B_0 parallel to B_1B_2 (see Figure 4)

$$g_2:\quad \mathbf{b}_0 \times (\mathbf{b}_2-\mathbf{b}_1) = \begin{pmatrix} 1 \\ b_{0x} \\ b_{0y} \end{pmatrix} \times \begin{pmatrix} 0 \\ b_{2x}-b_{1x} \\ b_{2y}-b_{1y} \end{pmatrix} = \begin{pmatrix} det(\mathbf{b}_0,\mathbf{b}_2-\mathbf{b}_1) \\ b_{1y}-b_{2y} \\ b_{2x}-b_{1x} \end{pmatrix}.$$

We obtain an analogue equation for the line g_1 (through A_0 parallel A_1A_2) and for g_3 (through C_0 parallel C_1C_2). The concurrence of the three lines g_1, g_2 and g_3 is equivalent to

$$det\, G = 0 \text{ with } G = \begin{pmatrix} (\mathbf{a}_2-\mathbf{a}_1)^\perp & det(\mathbf{a}_0,\mathbf{a}_2-\mathbf{a}_1) \\ (\mathbf{b}_2-\mathbf{b}_1)^\perp & det(\mathbf{b}_0,\mathbf{b}_2-\mathbf{b}_1) \\ (\mathbf{c}_2-\mathbf{c}_1)^\perp & det(\mathbf{c}_0,\mathbf{c}_2-\mathbf{c}_1) \end{pmatrix}$$

If the bars of the three arms already concurrent, that means, if $det\, M = 0$ with the matrix M of eq. (11), then the singularity of the matrix G is equivalent to that of $G - M$, since due to the coincidence of the first two columns of G and M we have $det\,(G - M) = det\, G - det\, M$. In detail, we obtain

$$G - M = \begin{pmatrix} (\mathbf{a}_2-\mathbf{a}_1)^\perp & det(\mathbf{a}_0,\mathbf{a}_2-\mathbf{a}_1) - det(\mathbf{a}_1,\mathbf{a}_2) \\ (\mathbf{b}_2-\mathbf{b}_1)^\perp & det(\mathbf{b}_0,\mathbf{b}_2-\mathbf{b}_1) - det(\mathbf{b}_1,\mathbf{b}_2) \\ (\mathbf{c}_2-\mathbf{c}_1)^\perp & det(\mathbf{c}_0,\mathbf{c}_2-\mathbf{c}_1) - det(\mathbf{c}_1,\mathbf{c}_2) \end{pmatrix}$$

Therein - we write again the second row-

$$\begin{aligned} det\,(\mathbf{b}_0,\mathbf{b}_2-\mathbf{b}_1) - det(\mathbf{b}_1,\mathbf{b}_2) &= det\,(\mathbf{b}_0,\mathbf{b}_2-\mathbf{b}_1) - det\,(\mathbf{b}_1,\mathbf{b}_2-\mathbf{b}_1) \\ &= det\,(\mathbf{b}_0-\mathbf{b}_1,\mathbf{b}_2-\mathbf{b}_1) \\ &= det\,(\mathbf{b}_2-\mathbf{b}_1,\mathbf{b}_1-\mathbf{b}_0). \end{aligned}$$

Hence, under the condition $det\, M = 0$ we obtain,

$$det\, G = 0 \Leftrightarrow \mathrm{rg}(M_{ext}) < 3.$$

It can be verified that this is also true for indirect mechanisms if the line g_2 parallel to B_1B_2 passes through the reflection of B_0 w.r.t. B_1 (for details see Can, 2012).

Theorem 6. a. Any pose of the mechanism is singular if and only if the three lines spanned by the arms A_1A_2, B_1B_2 and C_1C_2 are concurrent.

b. Any pose of the mechanism is twice singular if and only if the lines A_1A_2, B_1B_2 and C_1C_2 meet at a point G and the three lines g_1, g_2 and g_3, as explained before, meet at a common point $\bar{G}$.

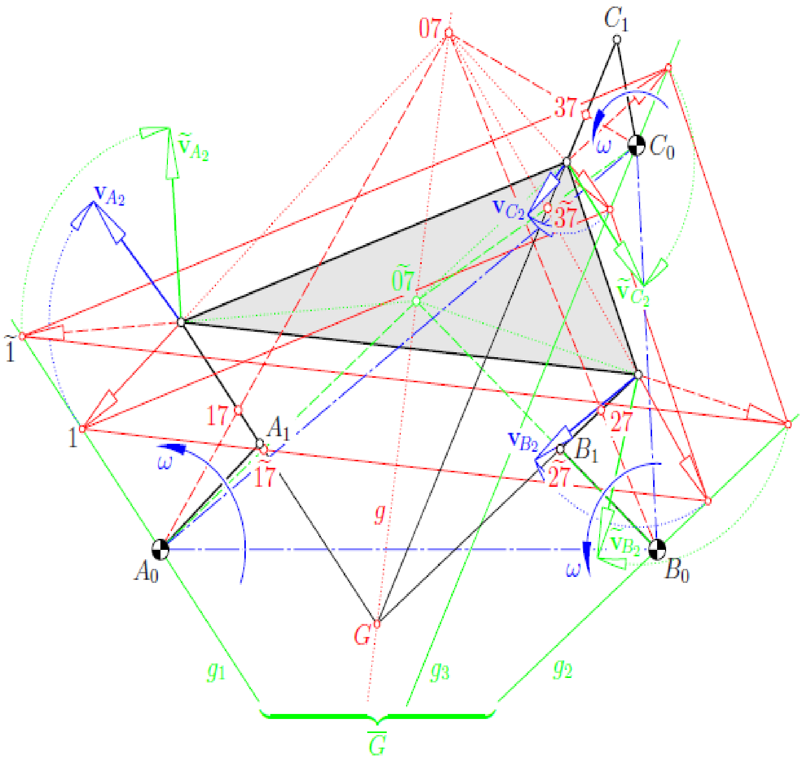

Figure 4. Twice singular pose of a direct mechanism (Theorem 6). All possible relative poles *07* of Σ_7/Σ_0 are located on the line *g,* which connects *G* and $\bar{G}$. (Can, 2012)

Theorem 7. The moving system Σ_7 has a still stand if and only if each arm is aligned with the neighboring crank and these three lines are not concurrent (see Figure 5).

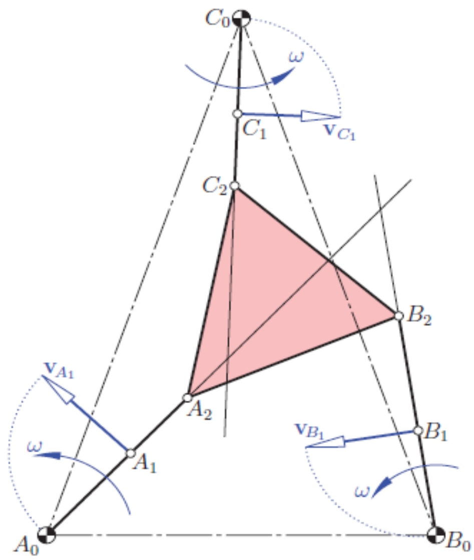

Figure 5. The moving plane has instantaneously still stand despite of $\omega \neq 0$.

Proof. By virtue of eq. (7), has a still stand is characterized by $(x_1, x_2, x_3) = (0, 0, 0)$. In this case the system (11) of equations M **x** = **s** has only the trivial solution. Therefore, this system must be homogeneous with rk $M = 3$. Each entry in the absolute column -like det ($\mathbf{b}_2$ - $\mathbf{b}_1$, $\mathbf{b}_1$ - $\mathbf{b}_0$) - vanishes if and only if each driving crank is aligned with the subsequent arm.

ANALYTIC ACCELERATION ANALYSIS OF THE MECHANISM

The velocity analysis led to the system (11) of linear equations for the unknowns (x_1, x_2, x_3), which define the vector $\mathbf{v}_0$ and the skew-symmetric matrix S in eq. (7).

By Theorem 3, the vehicular acceleration of any point attached to Σ_7 is

$$\mathbf{a_x} = \mathbf{a_0} + \left(\dot{S} + S^2\right)\mathbf{x}_0,$$

where $\mathbf{x_0}$ is the fixed coordinate of the point under consideration. We set up

$$\mathbf{a_0} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ and } \dot{S} = \begin{pmatrix} 0 & y_3 \\ -y_3 & 0 \end{pmatrix}. \tag{15}$$

Thus, we obtain for vertices of the moving triangle in Σ_7 the acceleration vectors

$$\mathbf{a_{a_2}} = \mathbf{a_0} + \left(\dot{S} + S^2\right)\mathbf{a}_2$$
$$\mathbf{a_{b_2}} = \mathbf{a_0} + \left(\dot{S} + S^2\right)\mathbf{b}_2$$
$$\mathbf{a_{c_2}} = \mathbf{a_0} + \left(\dot{S} + S^2\right)\mathbf{c}_2$$

where

$$\left(\dot{S} + S^2\right) = \begin{pmatrix} -x_3^2 & y_3 \\ -y_3 & -x_3^2 \end{pmatrix} \tag{16}$$

For a constant driving speed ω of the cranks we obtain as accelerations of their endpoints

$$\mathbf{a_{a_1}} = -\omega^2(\mathbf{a}_1 - \mathbf{a}_0)$$
$$\mathbf{a_{b_1}} = -\omega^2(\mathbf{b}_1 - \mathbf{b}_0)$$
$$\mathbf{a_{a_1}} = -\omega^2(\mathbf{c}_1 - \mathbf{c}_0).$$

According to theorem 4 - *we write again the second equation –*

$$\mathbf{a_{b_2}} - \mathbf{a_{b_1}} = \mathbf{a^n_{b_2b_1}} + \mathbf{a^t_{b_2b_1}}$$

where

$$\mathbf{a^n_{b_2b_1}} = -\frac{\|\mathbf{v_{b_2b_1}}\|^2}{\|(\mathbf{b}_2 - \mathbf{b}_1)\|^2}(\mathbf{b}_2 - \mathbf{b}_1).$$

The vector $\mathbf{a^t_{b_2b_1}}$ is orthogonal to the arm $\mathbf{b}_2 - \mathbf{b}_1$. These results in

$$\left(\mathbf{a_{b_2}} - \mathbf{a_{b_1}}\right).(\mathbf{b}_2 - \mathbf{b}_1) = \mathbf{a^n_{b_2b_1}}.(\mathbf{b}_2 - \mathbf{b}_1) = -\mathbf{v^2_{b_2b_1}} = -\left(\mathbf{v_{b_2}} - \mathbf{v_{b_1}}\right)^2.$$

After substituting the values from the equations (7), (8), (9) and (16) we obtain

$$\left(\mathbf{a_0} + \left(\dot{S} + S^2\right)\mathbf{b}_2 + \omega^2(\mathbf{b}_1 - \mathbf{b}_0)\right).(\mathbf{b}_2 - \mathbf{b}_1) = -\left(\mathbf{v_0} + S\mathbf{b}_2 \pm \omega D(\mathbf{b}_1 - \mathbf{b}_0)\right)^2.$$

This is a linear equation with the unknowns' y_1, y_2, and y_3. By virtue of eq. (12), it can be rewritten in matrix form as

$$(\mathbf{b}_2 - \mathbf{b}_1)^{\mathrm{T}}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + y_3\, det(\mathbf{b}_1, \mathbf{b}_2) - x_3^2\mathbf{b}_2^{\mathrm{T}}(\mathbf{b}_2 - \mathbf{b}_1) + \omega^2(\mathbf{b}_1 - \mathbf{b}_0)^{\mathrm{T}}(\mathbf{b}_2 - \mathbf{b}_1)$$
$$= -\left(\mathbf{v_0} + S\mathbf{b}_2 \pm \omega D(\mathbf{b}_1 - \mathbf{b}_0)\right)^{\mathrm{T}}\left(\mathbf{v_0} + S\mathbf{b}_2 \pm \omega D(\mathbf{b}_1 - \mathbf{b}_0)\right).$$

Furthermore, because of $S^{\mathrm{T}}S = x_3^2 I_2$, $D^{\mathrm{T}}D = I_2$ and $S^{\mathrm{T}}D = -x_3 I_2 = D^{\mathrm{T}}S$ follows

$$(\mathbf{b}_1 - \mathbf{b}_2).\mathbf{a_0} + y_3\, det(\mathbf{b}_1, \mathbf{b}_2) = -x_3^2\mathbf{b}_2(\mathbf{b}_2 - \mathbf{b}_1) + \omega^2(\mathbf{b}_1 - \mathbf{b}_0)(\mathbf{b}_2 - \mathbf{b}_1) + [\mathbf{v_0}^2 + x_3^2\mathbf{b}_2^2 + \omega^2(\mathbf{b}_1 - \mathbf{b}_0)^2 + 2x_3\, det(\mathbf{v_0}, \mathbf{b}_2) \pm 2\omega\, det(\mathbf{v_0}, \mathbf{b}_1 - \mathbf{b}_0) \pm 2\omega\, x_3\, \mathbf{b}_2.(\mathbf{b}_1 - \mathbf{b}_0)].$$

We summarize all as fallows

Theorem 8. The instantaneous acceleration distribution of the system Σ_7, i.e., the components (y_1, y_2) of $\mathbf{a_0}$ and the non-trivial entry y_3 of the skew-symmetric matrix $\dot{S}$, solve the linear system $M\,\mathbf{y} = \mathbf{t}$ with the coefficient matrix M of eq. (11). The absolute column is reads

$$\mathbf{t} = \begin{pmatrix} \mathbf{v_0}^2 + x_3^2\mathbf{a}_1\mathbf{a}_2 + 2x_3\, det(\mathbf{v_0}, \mathbf{a}_2) + 2\omega\, x_3\, \mathbf{a}_2.(\mathbf{a}_1 - \mathbf{a}_0) + 2\omega\, det(\mathbf{v_0}, \mathbf{a}_1 - \mathbf{a}_0) + \omega^2(\mathbf{a}_1 - \mathbf{a}_0)(\mathbf{a}_2 - \mathbf{a}_0) \\ \mathbf{v_0}^2 + x_3^2\mathbf{b}_1\mathbf{b}_2 + 2x_3\, det(\mathbf{v_0}, \mathbf{b}_2) \pm 2\omega\, x_3\, \mathbf{b}_2.(\mathbf{b}_1 - \mathbf{b}_0) \pm 2\omega\, det(\mathbf{v_0}, \mathbf{b_1} - \mathbf{b}_0) + \omega^2(\mathbf{b}_1 - \mathbf{b}_0)(\mathbf{b}_2 - \mathbf{b}_0) \\ \mathbf{v_0}^2 + x_3^2\mathbf{c}_1\mathbf{c}_2 + 2x_3\, det(\mathbf{v_0}, \mathbf{c}_2) + 2\omega\, x_3\, \mathbf{c}_2.(\mathbf{c}_1 - \mathbf{c}_0) + 2\omega\, det(\mathbf{v_0}, \mathbf{c}_1 - \mathbf{c}_0) + \omega^2(\mathbf{c}_1 - \mathbf{c}_0)(\mathbf{c}_2 - \mathbf{c}_0) \end{pmatrix}. \quad (17)$$

Theorem 9. If at a still stand of the moving plane Σ_7 with three collinear point triples (A_0, A_1, A_2), (B_0, B_1, B_2) and (C_0, C_1, C_2) also all accelerations vanish, then the pose has a permanent still stand (see Figure 6).

Proof. A still stand of Σ_7 with vanishing velocity vectors is characterized by $\mathbf{v_0} = \mathbf{0}$ and $x_3 = 0$. Under this assumption, in the entries of the absolute column **t** only the respective last summands are remaining. Therefore, we obtain zero accelerations if and only if beside det($\mathbf{a}_1 - \mathbf{a}_0$, $\mathbf{a}_2 - \mathbf{a}_0$) = ... = 0 we also have ($\mathbf{a}_1 - \mathbf{a}_0$). ($\mathbf{a}_2 - \mathbf{a}_0$) = ... = 0. The first condition means collinearity, the second orthogonality. Hence, the unique solution is $\mathbf{a}_2 = \mathbf{a}_0$, $\mathbf{b}_2 = \mathbf{b}_0$ and $\mathbf{c}_2 = \mathbf{c}_0$. This results in the permanent still stand as displayed
in Figure 6.

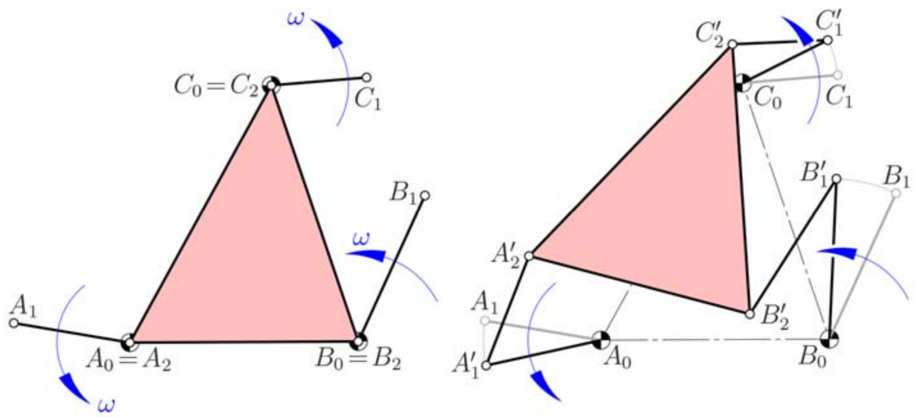

Figure 6. Vanishing velocities and accelerations for all points of Σ_7 result in a permanent still stand of the system. Nevertheless, this pose of Σ_7 admits a bifurcation to a constrained motion.

Remark 4. It is not surprising that in the system of linear equations M **y** = **t**, which defines the accelerations, the same coefficient matrix M shows up as for the velocities. We know this from the definition of infinitesimal flexibility of higher order (see Sabitov, 1992).

With regard to the set of solutions of the linear system addresses in Theorem 8, again the three cases **(i), (ii)** and **(iii)** can be distinguished, like for Theorem 5. Because of the identity of the coefficient matrices in equations (11) and (17), the case **(i)**, i.e., rk (M) < 3, is still characterized by the concurrence of the lines spanned by the three arms. However, for the cases **(ii)** and **(iii)** no simple geometric characterizations can be expected, since the unknowns x_1, x_2, x_3 of (11) show up on the right-hand side of eq. (17) in linear form and as squares.

CONCLUSIONS

This kind of mechanisms, i.e., planar parallel 3-RRR robots with three synchronously driven cranks, offer a broad variety of different periodical constrained motions, including a permanent still stand. On the other hand, these constrained motions can pass through poses with very special properties, as has been demonstrated before. We conclude with another strategy for obtaining special poses:

Let us, temporarily, remove the crank C_0C_1 and the arm C_1C_2. Then the moving triangle in Σ_7 can perform a two-parameter motion. Within this two-parameter motion, all constrained motions passing through a fixed pose have their instantaneous pole on a line, the *pole axis* (Blaschke and Müller, 1956 and Stachel, 1979). (In Figure 7 this pole axis is marked in thick blue). During this two-parameter motion the points X attached to Σ_7 sweep out two-dimensional regions. If any pose of X happens to lie on the boundary of this region then X must also lie on the pole axis of this pose, which is orthogonal to the boundary at X. In Figure 7 point C_2 is specified on the corresponding pole axis, but at the same time also on the boundary of the circular region, which the endpoint C_2 of the dyad $C_0C_1C_2$ sweeps out. By virtue of Theorem 6, this pose of the 3-RRR-mechanism is twice singular. When the region traced by C_2 under the two-parameter motion, as explained before, shares with the circular disk centered at C_0 only this boundary point, then this pose is isolated and only infinitesimally movable (see Figure 7). Otherwise it enables a bifurcation between two constrained motions. This can be figured out with the help of the (t, φ_{70})-diagram in

Figure 8, which covers the special choice $C_0 = 04$. With regard to this diagram, t is the driving angle of the cranks, and φ is the angle of rotation of Σ_7.
The bifurcation shows up at $t = 64.59^0$.

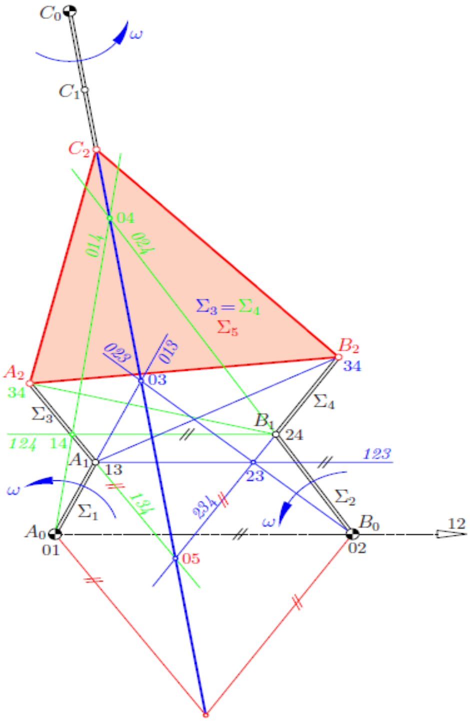

Figure 7. This is an isolated, only infinitesimally movable pose, which is twice singular.

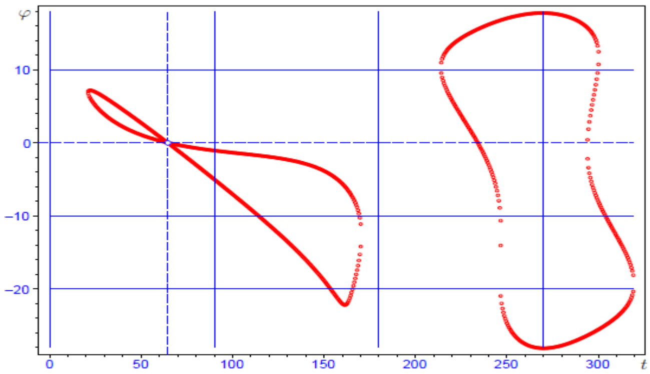

Figure 8. When in addition to the choice of C_2 on the instant pole axis (like in Figure 7) the anchor point C_0 is specified as $C_0 = 04$, then this pose is no more isolated, but enables a bifurcation.

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