SPECTRAL PROPERTIES OF NON-SELF-ADJOINT ELLIPTIC DIFFERENTIAL OPERATORS

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ABSTRACT
The study of non-self-adjoint differential operators is a historical issue. Before and until now, most studies of operator theory have been about self-adjoint operators. But non-self-adjoint operators have recently found many applications in other sciences. These operators have received much attention in thermodynamics and quantum mechanics. Because there is no general spectral theory for these operators, it is more difficult to study these operators than the self-adjoint type. The spectrum of these operators is usually unstable and their resolvent is unpredictable. In this paper, a second-order non-self-attached differential operator is considered and its spectral properties and solvent estimation are studied. This operator is much more common than second-tier operators such as Storm-Liouville and Schrodinger.
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1. Introduction

Let $\Omega \subset \mathrm{R}^n$ be a bounded domain with smooth boundary i.e. $\partial\Omega \in C^\infty$. Let $Au(x) = -\sum_{i,j=1}^{n}\left(\omega(x)q(x)u'_{x_i}(x)\right)'_{x_j}$ defined in the space $H_\ell = L_2(\Omega)^\ell$.

Here $q(x) \in C^2\left([0,1], End\,\mathbf{C}^\ell\right)$ and $\omega \in C^1\left([0,1]\right)$ is a non negative function. Then

A is a non-self-adjoint differential operator. Let the weighted Sobolev space $\mathrm{H}_\ell = W^2_{2,\omega}(\Omega)^\ell$ as the space of vector functions $u(x) = \left(u_1(x), u_2(x), \ldots, u_\ell(x)\right)$ defined on Ω with finite norm

$$\|u\|_s = \left(\sum_{i=1}^{n}\int_\Omega \omega(x)\left|u'_{x_i}(x)\right|^2_{C^\ell}dx + \int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}}$$

To find the spectral properties of this operator, assume that

$$D(A) = \left\{u \in \overset{\circ}{\mathrm{H}}_\ell \cap W^2_{2,loc}(\Omega)^\ell : \sum_{i,j=1}^{n}\left(\omega(x)q(x)u'_{x_i}(x)\right)'_{x_j} \in H_\ell\right\}$$

Let for all $x \in \Omega$, $q(x)$ is diagonal matrix, that is assume that $q(x)$ has ℓ simple eigenvalues $\mu_1(x), \mu_2(x), \ldots, \mu_\ell(x)$ in the complex plane in the following way:

$$\mu_1(x), \mu_2(x), \ldots, \mu_r(x) \in \mathrm{R}^+ \text{ (Then } \arg\mu_j(x) = 0 \quad for \quad j = 1,2,\ldots,r \text{) and}$$

$$\mu_j(x) \in S, j = r+1, r+2, \ldots, \ell \quad \text{where } S = \left\{z \in C : \frac{\pi}{2} < |\arg z| < \pi\right\}$$

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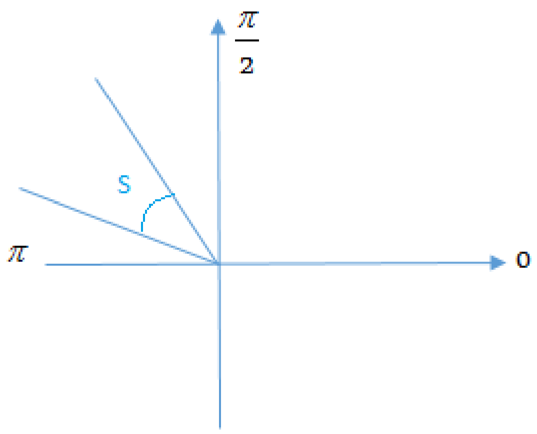
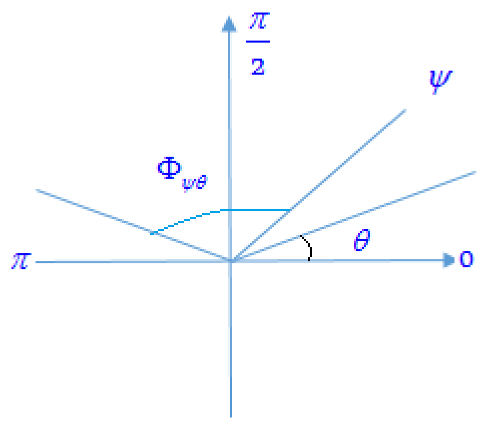


Let the angle $\Phi_{\psi\theta} = \{z \in C : |\arg z| > \psi\}$, $\theta < \frac{\pi}{2}, \psi \in \left(\theta, \frac{\pi}{2}\right)$



One of the things that has been done in this article is to resolvent estimate of this operator. That is, it is proved that for sufficiently large in modulus $\lambda \in \Phi_{\psi\theta}$ the inverse operator $(A - \lambda I)^{-1}$ exist and is continuous.

Lemma 1.1.

There is a matrix function $a(x) \in C^2([0,1], End\,\mathbf{C}^{\ell})$ such that $a^{-1}(x) \in C^2([0,1], End\,\mathbf{C}^{\ell})$ and

$q(x) = a(x)\Lambda(x)a^{-1}(x)$. Here $\Lambda(x)$ is a diagonal matrix, i.e.

$$\Lambda(x) = diag\{\mu_1(x), \mu_2(x), \ldots, \mu_{\ell}(x)\}, \mu_j(x) \in C^2(\Omega), j = 1, 2, \ldots, \ell$$

2. On the resolvent of A

Lemma 2.1.

For $\lambda \in \Phi_{\psi\theta}$ there exist $\gamma \in (-\pi, \pi]$ such that for some positive number c we have:

$$c|\lambda| \leq -\mathrm{Re}\{e^{i\gamma}\lambda\}$$

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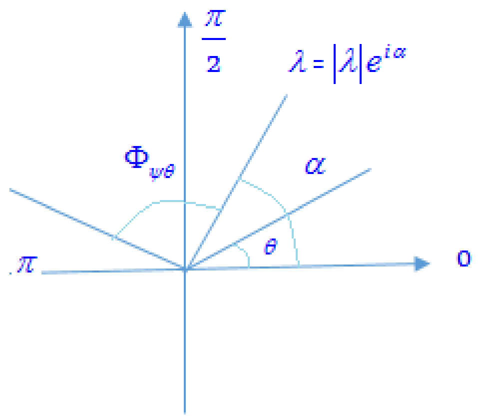
Proof:

Let $\lambda \in \Phi_{\psi\theta}$ and $\lambda = |\lambda| e^{i\alpha}, \alpha > \theta, \alpha < \dfrac{\pi}{2}$



Let γ such that $\dfrac{\pi}{2} < \gamma + \alpha < \pi$ therefore there exist $\varepsilon > 0$ such that $\gamma = \dfrac{\pi}{2} - \alpha + \varepsilon$

$\cos(\gamma + \alpha) = \cos\left(\dfrac{\pi}{2} + \varepsilon\right)$ and $\cos\left(\dfrac{\pi}{2} + \varepsilon\right) < 0$. But $\operatorname{Re}\{e^{i\gamma}\lambda\} = \operatorname{Re}\left\{|\lambda| e^{i(\gamma+\alpha)}\right\} = |\lambda| \cos\left(\dfrac{\pi}{2} + \varepsilon\right)$

Then $\operatorname{Re}\left(e^{i\gamma}\lambda\right) < 0$

$$\operatorname{Re}\left(e^{i\gamma}\lambda\right) < 0 \Rightarrow \exists c > 0 : c < -\operatorname{Re}\left(e^{i\gamma}\lambda\right) \Rightarrow$$

$$c|\lambda| \leq -\operatorname{Re}\left(e^{i\gamma}|\lambda| e^{i\alpha}\right)$$

$$\Rightarrow c|\lambda| \leq -\operatorname{Re}\left(e^{i\gamma}\lambda\right)$$

Theorem 2.2.

Let A be the operator that defined in above. For sufficiently large in modulus $\lambda \in \Phi_{\psi\theta}$ the inverse operator $(A - \lambda I)^{-1}$ exist and is continuous. In the other word for sufficiently large in modulus $\lambda \in \Phi_{\psi\theta}$ there exist $M_{\psi\theta}, C_{\psi\theta} > 0$, such that the following estimate is valid, that is the resolvent of A exist,

$$\left\|(A - \lambda I)^{-1}\right\| \leq M_{\psi\theta} |\lambda|^{-1}, |\lambda| \geq C_{\psi\theta}$$

Proof. Let the operator $Au(x) = -\sum_{i,j=1}^{n} \left(\omega(x) q(x) u'_{x_i}(x)\right)'_{x_j}$ where

$u(x) = \left(u_1(x), u_2(x), \ldots, u_\ell(x)\right), x \in \Omega \subset \mathbf{R}^n$ And let the weighted Sobolev space $\mathrm{H}_\ell = W_{2,\alpha}^{2}(\Omega)^\ell \subset H_\ell$ with finite norm

$$\|u\|_s = \left(\sum_{i=1}^{n} \int_{\Omega} \omega(x) \left| u'_{x_i}(x) \right|^2_{C^\ell} dx + \int_{\Omega} |u(x)|^2 dx \right)^{\frac{1}{2}}$$

By this norm, we defined $\overset{\circ}{\mathrm{H}}_\ell \subset \mathrm{H}_\ell$ such that $\overset{\circ}{\mathrm{H}}_\ell$ =closure of $C_0^\infty(\Omega)^\ell$ in H_ℓ.

And then the domain of A as fallow

$$D(A) = \left\{ u \in \overset{\circ}{\mathrm{H}}_\ell \cap W^2_{2,loc}(\Omega)^\ell : \sum_{i,j=1}^{n} \left(\omega(x) q(x) u'_{x_i}(x) \right)'_{x_j} \in H_\ell \right\}$$

Let $q(x)$ to be the matrix function with ℓ distinct simple eigenvalues $\mu_1(x), \mu_2(x), ..., \mu_\ell(x) \in C^2(\Omega)$ and this eigenvalues arranged in the complex plane in the fallowing way:

$\exists r \in \{1,2,...,\ell\}$ Such that for this r: $\mu_j(x) \in \mathbf{R}^+$ that is $\arg \mu_j(x) = 0$ for (j=1, 2,… , r)

And for $j = r+1, r+2, \ldots, \ell$ we have: $\mu_j(x) \in \Phi_{\psi\theta}$ where

$\Phi_{\psi\theta} = \{ z \in C : |\arg z| > \psi \}, \theta < \frac{\pi}{2}, \psi \in \left(\theta, \frac{\pi}{2}\right)$. Now we defined the operators

$$A_j u(x) = -\sum_{i,j=1}^{n} \left(\omega(x) \mu_j(x) u'_{x_i}(x) \right)'_{x_j}, j = r+1, r+2, ..., n,$$

and

$$D(A_j) = \left\{ u \in \overset{\circ}{\mathrm{H}}_\ell \cap W^2_{2,loc}(\Omega)^\ell : \sum_{i,j=1}^{n} \left(\omega(x) \mu_j(x) u'_{x_i}(x) \right)'_{x_j} \in H_\ell = H \right\}.$$

Now let $I(u) = \sum_{i=1}^{n} \int_\Omega \omega(x) \left| u'_{x_i}(x) \right|^2 dx$. For every $S = (s_1, ..., s_n) \in C^n$ we have uniformly elliptic condition $|S|^2 \le c \operatorname{Re} \sum_{i,j=1}^{n} s_i \bar{s}_j$ if $s_i = u'_{x_i}, s_j = u'_{x_j}$ then

$$|S|^2 = \sum_{i=1}^{n} |s_i|^2 \le c \operatorname{Re} \sum_{i,j=1}^{n} s_i \bar{s}_j \le c \operatorname{Re} \sum_{i,j=1}^{n} u'_{x_i} \bar{u}'_{x_j},$$

and so

$$I(u) = \sum_{i=1}^{n} \int_\Omega \omega(x) \left| u'_{x_i}(x) \right|^2 dx \le c \operatorname{Re} \left\{ e^{i\gamma} \sum_{i,j=1}^{n} \omega(x) \mu_k(x) u'_{x_i} \bar{u}'_{x_j} \right\}$$

$$\Rightarrow I(u) \le c \operatorname{Re} e^{i\gamma} \left\langle -\sum_{i,j=1}^{n} \left(\omega(x) \mu_k(x) u'_{x_i}(x) \right)'_{x_j}, u(x) \right\rangle$$

But we have $c|\lambda| \le -\operatorname{Re}\left(e^{i\gamma}\lambda\right)$ then

$$I(u)+|\lambda||u|^2 \le c\,\mathrm{Re}\langle e^{i\gamma}(A_k-\lambda I)u,u\rangle$$

So we have:

$$I(u)+|\lambda|\,\|u\|^2 \le c\,\mathrm{Re}\,\|u\|\,\|(A_k-\lambda I)(u)\|$$

And so

$$|\lambda|\,\|u\|^2 \le c\,\|u\|\,\|(A_k-\lambda I)(u)\| \Rightarrow |\lambda||u| \le c\,\|(A_k-\lambda I)(u)\|.$$

By this relation we conclude that $A_k-\lambda I$ is a one to one operator and $\left\|(A_k-\lambda I)^{-1}\right\| \le \frac{1}{c}|\lambda|^{-1} = M\,|\lambda|^{-1}$, similar inequalities apply to operators $A_k, k = 1,2,..,r$. Because these operators are self-adjoint. So this estimate hold for linear operator A. see [1,2,4]

3. On the spectrum of A

Theorem 3.1.

The operator A has a discrete spectrum for any $\theta < \frac{\pi}{2}, \psi \in \left(\theta, \frac{\pi}{2}\right)$

Proof.

If $\lambda \in \rho(A)$ then $(A-\lambda I)^{-1}$ is compact, by Relish's theorem. Therefore, the Riesz-Schaud theory of compact operators implies that the spectrum of $(A-\lambda I)^{-1}$ consists only of eigenvalues of finite multiplicity, whose only possible limit point is the number 0, which is also in the spectrum of $(A-\lambda I)^{-1}$. This implies that the spectrum of A is itself discrete, (see [3]) so every complex number in the spectrum is an eigenvalue. And the eigenvalues of A have finite multiplicity.

Theorem 3.2.

Denote by $\lambda_1, \lambda_2, \lambda_3, \ldots$ the eigenvalues of A then: $\arg\lambda_j \to 0 \quad (j \to +\infty)$

Proof.

Because $\{\arg\lambda_j : j = 1,2,3,\ldots\}$ is a bounded set so it has a limit point. Zero is the only limit point in this set. If it has a nonzero limit point $\{\arg\lambda_{j_k}\}$ sequence -, then there is a convergent sub$[-\phi,\phi]$ in the interval ϕ_1 in this set that converges to ϕ_1 .Suppose S is a closed sector with zero vertices containing radius $\Upsilon = \{z \in \mathbf{C} : \arg z = \phi_1\}$ in the complex plane.now let N_1 be natural number such that for every there is a natural numberso $\lambda_j \to \infty$ .but $k \ge N_1; \lambda_{j_k} \in S$

N_2 Such that $|\lambda_{j_k}| > C_{\theta_1}$ for $k \ge N_2$ .if $k > \max\{N_1, N_2\}$ then $\lambda_{j_k} \in S, |\lambda_{j_k}| > C_{\theta_1}$ so theorem3.6 implies that λ_{j_k} is not eigenvalues that is a contradiction.Then we have: $\arg\lambda_j \to 0 \quad (j \to +\infty)$

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